Backpropagation in CNN

11785 Introduction to Deep Learning (Fall 2022) - Recitation 5

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Backpropagation in CNNs

- In the backward pass, we have the derivatives of the loss w.r.t the next layer
- Mainly need to calculate the derivatives of
  - Input of conv layer
  - Weights/Filter of conv layer

Backprop in CNN with single stride

- Let us consider an example with **stride 1** both in horizontal and vertical direction
- We have Input X and Filter as shown

X_{11}	X_{12}	X_{13}
X_{21}	X_{22}	X_{23}
X_{31}	X_{32}	X_{33}

Input X

F_{11}	F_{12}
F_{21}	F_{22}

Filter F

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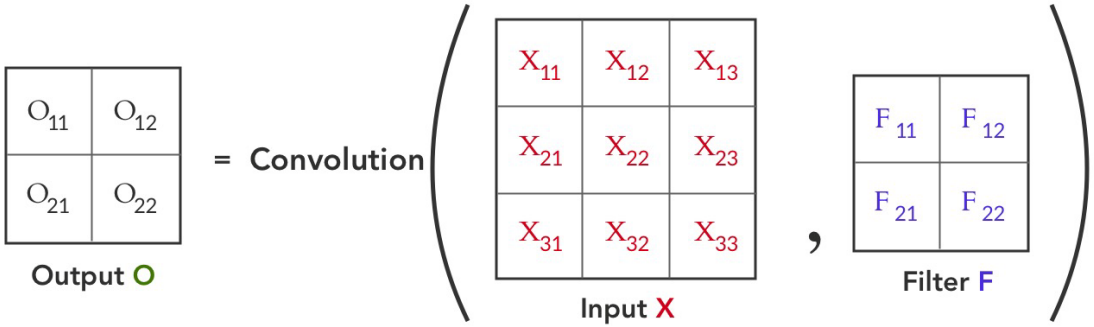
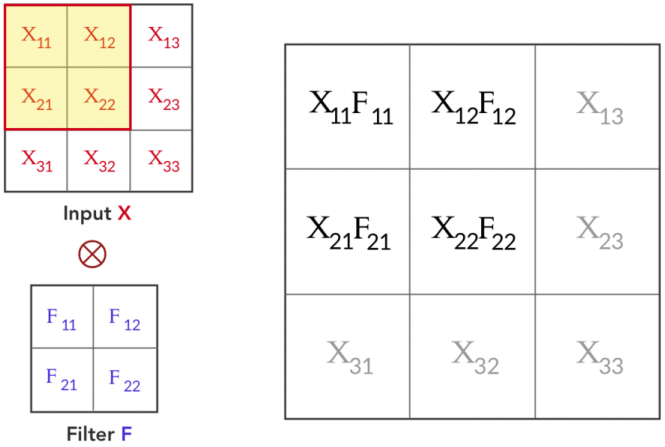
Forward Pass

- Convolution between Input X and Filter F results in an output O as shown:



Forward Pass

- Convolution between Input X and Filter F results in an output O as shown:



$$O_{11} = X_{11}F_{11} + X_{12}F_{12} + X_{21}F_{21} + X_{22}F_{22}$$

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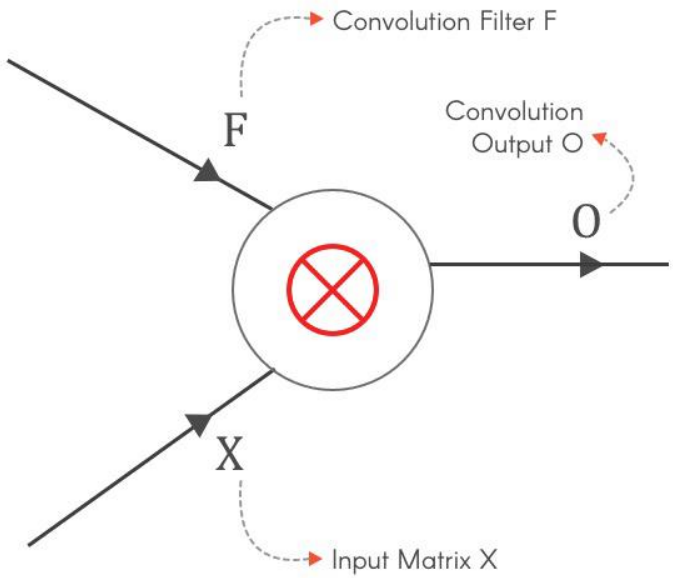
Computing Derivatives

- We want to calculate derivatives of output O w.r.t. Input X and Filter F



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Derivative of Loss w.r.t Filter

- We can apply chain rule as follows

O: affine output of current layer

$$\frac{\partial L}{\partial F} = \frac{\partial L}{\partial O} * \frac{\partial O}{\partial F}$$

For every element of F

$$\frac{\partial L}{\partial F_i} = \sum_{k=1}^{M} \frac{\partial L}{\partial O_k} * \frac{\partial O_k}{\partial F_i}$$

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Derivative of Loss w.r.t Filter

- We can expand the chain rule summation as follows:

For every element of F

$$\frac{\partial L}{\partial F_i} = \sum_{k=1}^{M} \frac{\partial L}{\partial O_k} * \frac{\partial O_k}{\partial F_i}$$

$$O_{11} = X_{11}F_{11} + X_{12}F_{12} + X_{21}F_{21} + X_{22}F_{22}$$

$$O_{12} = X_{12}F_{11} + X_{13}F_{12} + X_{22}F_{21} + X_{23}F_{22}$$

$$O_{21} = X_{21}F_{11} + X_{22}F_{12} + X_{31}F_{21} + X_{32}F_{22}$$

$$O_{22} = X_{22}F_{11} + X_{23}F_{12} + X_{32}F_{21} + X_{33}F_{22}$$

$$\frac{\partial L}{\partial F_{11}} = \frac{\partial L}{\partial O_{11}} * \frac{\partial O_{11}}{\partial F_{11}} + \frac{\partial L}{\partial O_{12}} * \frac{\partial O_{12}}{\partial F_{11}} + \frac{\partial L}{\partial O_{21}} * \frac{\partial O_{21}}{\partial F_{11}} + \frac{\partial L}{\partial O_{22}} * \frac{\partial O_{22}}{\partial F_{11}}$$

$$\frac{\partial L}{\partial F_{12}} = \frac{\partial L}{\partial O_{11}} * \frac{\partial O_{11}}{\partial F_{12}} + \frac{\partial L}{\partial O_{12}} * \frac{\partial O_{12}}{\partial F_{12}} + \frac{\partial L}{\partial O_{21}} * \frac{\partial O_{21}}{\partial F_{12}} + \frac{\partial L}{\partial O_{22}} * \frac{\partial O_{22}}{\partial F_{12}}$$

$$\frac{\partial L}{\partial F_{21}} = \frac{\partial L}{\partial O_{11}} * \frac{\partial O_{11}}{\partial F_{21}} + \frac{\partial L}{\partial O_{12}} * \frac{\partial O_{12}}{\partial F_{21}} + \frac{\partial L}{\partial O_{21}} * \frac{\partial O_{21}}{\partial F_{21}} + \frac{\partial L}{\partial O_{22}} * \frac{\partial O_{22}}{\partial F_{21}}$$

$$\frac{\partial L}{\partial F_{22}} = \frac{\partial L}{\partial O_{11}} * \frac{\partial O_{11}}{\partial F_{22}} + \frac{\partial L}{\partial O_{12}} * \frac{\partial O_{12}}{\partial F_{22}} + \frac{\partial L}{\partial O_{21}} * \frac{\partial O_{21}}{\partial F_{22}} + \frac{\partial L}{\partial O_{22}} * \frac{\partial O_{22}}{\partial F_{22}}$$

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Derivative of Loss w.r.t Filter

$$O_{11} = X_{11}F_{11} + X_{12}F_{12} + X_{21}F_{21} + X_{22}F_{22}$$

$$O_{12} = X_{12}F_{11} + X_{13}F_{12} + X_{22}F_{21} + X_{23}F_{22}$$

$$O_{21} = X_{21}F_{11} + X_{22}F_{12} + X_{31}F_{21} + X_{32}F_{22}$$

$$O_{22} = X_{22}F_{11} + X_{23}F_{12} + X_{32}F_{21} + X_{33}F_{22}$$

Forward Pass Equations

$$\frac{\partial L}{\partial F_{11}} = \frac{\partial L}{\partial O_{11}} * X_{11} + \frac{\partial L}{\partial O_{12}} * X_{12} + \frac{\partial L}{\partial O_{21}} * X_{21} + \frac{\partial L}{\partial O_{22}} * X_{22}$$

$$\frac{\partial L}{\partial F_{12}} = \frac{\partial L}{\partial O_{11}} * X_{12} + \frac{\partial L}{\partial O_{12}} * X_{13} + \frac{\partial L}{\partial O_{21}} * X_{22} + \frac{\partial L}{\partial O_{22}} * X_{23}$$

$$\frac{\partial L}{\partial F_{21}} = \frac{\partial L}{\partial O_{11}} * X_{21} + \frac{\partial L}{\partial O_{12}} * X_{22} + \frac{\partial L}{\partial O_{21}} * X_{31} + \frac{\partial L}{\partial O_{22}} * X_{32}$$

$$\frac{\partial L}{\partial F_{22}} = \frac{\partial L}{\partial O_{11}} * X_{22} + \frac{\partial L}{\partial O_{12}} * X_{23} + \frac{\partial L}{\partial O_{21}} * X_{32} + \frac{\partial L}{\partial O_{22}} * X_{33}$$

Backward Equations

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Derivative of Loss w.r.t Filter

If you look at this operation closely, it could be computed by convolving the input map with output affine derivative map

$$\begin{bmatrix} \frac{\partial L}{\partial F_{11}} & \frac{\partial L}{\partial F_{12}} \\ \frac{\partial L}{\partial F_{21}} & \frac{\partial L}{\partial F_{22}} \end{bmatrix} = \text{Convolution}\left(\begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix}, \begin{bmatrix} \frac{\partial L}{\partial O_{11}} & \frac{\partial L}{\partial O_{12}} \\ \frac{\partial L}{\partial O_{21}} & \frac{\partial L}{\partial O_{22}} \end{bmatrix} \right)$$

Can be viewed as convolution of input and affine derivative

$$\frac{\partial L}{\partial F_{11}} = \frac{\partial L}{\partial O_{11}} * X_{11} + \frac{\partial L}{\partial O_{12}} * X_{12} + \frac{\partial L}{\partial O_{21}} * X_{21} + \frac{\partial L}{\partial O_{22}} * X_{22}$$

$$\frac{\partial L}{\partial F_{12}} = \frac{\partial L}{\partial O_{11}} * X_{12} + \frac{\partial L}{\partial O_{12}} * X_{13} + \frac{\partial L}{\partial O_{21}} * X_{22} + \frac{\partial L}{\partial O_{22}} * X_{23}$$

$$\frac{\partial L}{\partial F_{21}} = \frac{\partial L}{\partial O_{11}} * X_{21} + \frac{\partial L}{\partial O_{12}} * X_{22} + \frac{\partial L}{\partial O_{21}} * X_{31} + \frac{\partial L}{\partial O_{22}} * X_{32}$$

$$\frac{\partial L}{\partial F_{22}} = \frac{\partial L}{\partial O_{11}} * X_{22} + \frac{\partial L}{\partial O_{12}} * X_{23} + \frac{\partial L}{\partial O_{21}} * X_{32} + \frac{\partial L}{\partial O_{22}} * X_{33}$$

Loss Derivative Equations

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Derivative of Loss w.r.t Input

- Simplifying for the local derivative we get

For every element of X_i

$$\frac{\partial L}{\partial X_i} = \sum_{k=1}^{M} \frac{\partial L}{\partial O_k} * \frac{\partial O_k}{\partial X_i}$$

Derivative of Loss w.r.t Input

$$O_{11} = X_{11}F_{11} + X_{12}F_{12} + X_{21}F_{21} + X_{22}F_{22}$$

$$O_{12} = X_{12}F_{11} + X_{13}F_{12} + X_{22}F_{21} + X_{23}F_{22}$$

$$O_{21} = X_{21}F_{11} + X_{22}F_{12} + X_{31}F_{21} + X_{32}F_{22}$$

$$O_{22} = X_{22}F_{11} + X_{23}F_{12} + X_{32}F_{21} + X_{33}F_{22}$$

Forward Pass Equations

$$\frac{\partial L}{\partial X_{21}} = \frac{\partial L}{\partial O_{11}} * F_{21} + \frac{\partial L}{\partial O_{21}} * F_{11}$$

$$\frac{\partial L}{\partial X_{22}} = \frac{\partial L}{\partial O_{11}} * F_{22} + \frac{\partial L}{\partial O_{12}} * F_{21} + \frac{\partial L}{\partial O_{21}} * F_{12} + \frac{\partial L}{\partial O_{22}} * F_{11}$$

$$\frac{\partial L}{\partial X_{23}} = \frac{\partial L}{\partial O_{12}} * F_{22} + \frac{\partial L}{\partial O_{22}} * F_{12}$$

Backward Equations

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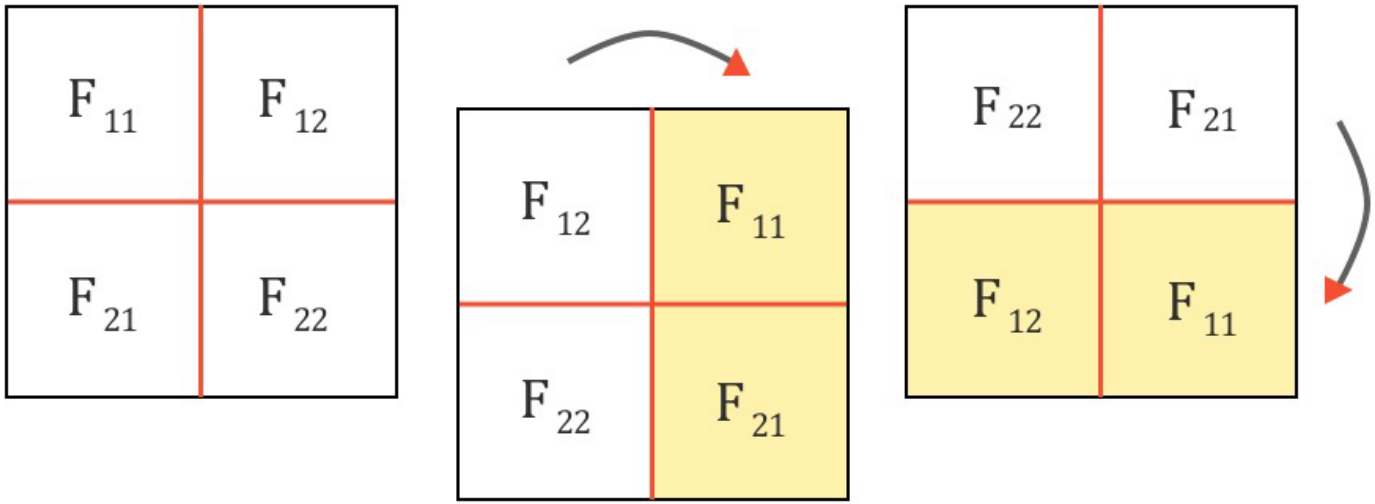
Derivative of Loss w.r.t Input

- **SImplifying computation:**
  - Flip the Filter horizontally and vertically



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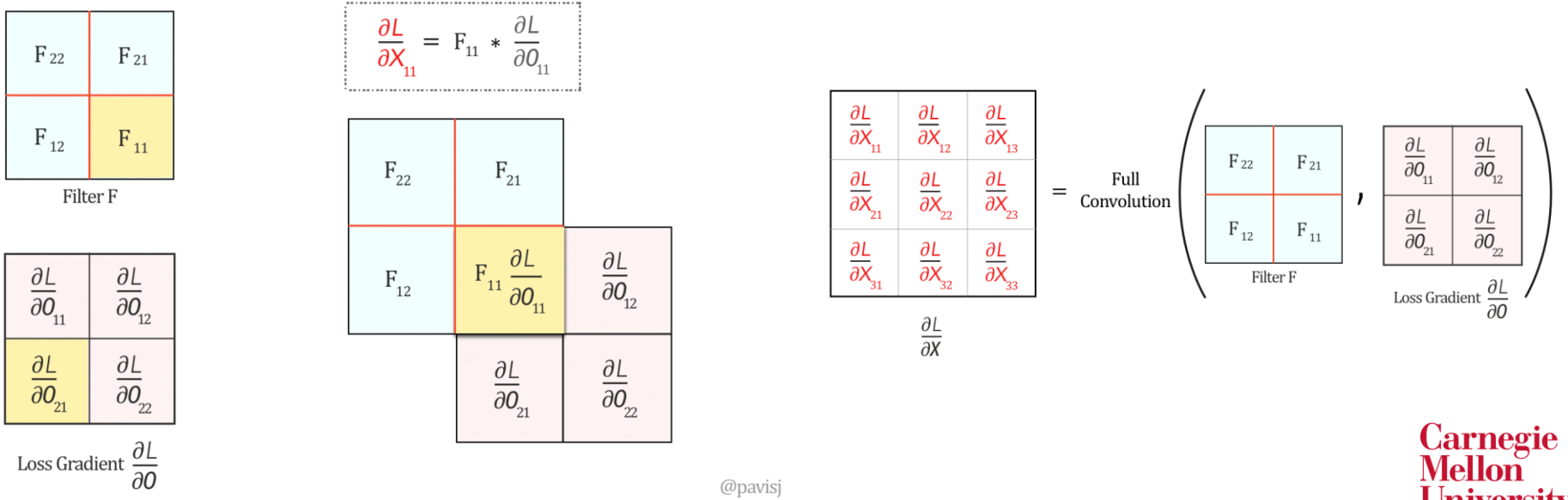
Derivative of Loss w.r.t Input

- **SImplifying computation:**
  - Flip the Filter horizontally and vertically
  - Perform Full Convolution of flipped filter with the affine derivatives



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Takeaway - Stride 1 BackProp

- Both the forward and backward pass of a convolution layer can be viewed as Convolution operations

$$\frac{\partial L}{\partial F} = \text{Convolution}\left(\text{Input } X\text{, Loss gradient } \frac{\partial L}{\partial O}\right)$$

$$\frac{\partial L}{\partial X} = \text{Full Convolution}\left(\text{180}^{\circ}\text{rotated Filter } F\text{, Loss Gradient } \frac{\partial L}{\partial O}\right)$$

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Backprop in CNN stride > 1

- Let us consider an example with **stride > 1** both in horizontal and vertical direction
- We have Input X and Filter as shown

W

H

x_{00}	x_{01}	x_{02}	x_{03}	x_{04}
x_{10}	x_{11}	x_{12}	x_{13}	x_{14}
x_{20}	x_{21}	x_{22}	x_{23}	x_{24}
x_{30}	x_{31}	x_{32}	x_{33}	x_{34}
x_{40}	x_{41}	x_{42}	x_{43}	x_{44}

Input activations
Input channels C = 1, number of images N = 1,
Image height H = 5, width = 5

S

R

f_{00}	f_{01}	f_{02}
f_{10}	f_{11}	f_{12}
f_{20}	f_{21}	f_{22}

Filter (aka kernel)
Input channels C = 1, number of filters K = 1,
Filter height R = 3, width S = 3,
stride_R = stride_S = 2

Q

P

y_{00}	y_{01}
y_{10}	y_{11}

Output
Output channels K = 1, number of outputs N = 1,
Output height P = 2, width Q = 2

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Forward Pass for stride > 1

- Let us consider an example with **stride > 1** both in horizontal and vertical direction
- We have Input X and Filter as shown

$x_{00}f_{00}$	$x_{01}f_{01}$	$x_{02}f_{02}$	x_{03}	x_{04}
$x_{10}f_{10}$	$x_{11}f_{11}$	$x_{12}f_{12}$	x_{13}	x_{14}
$x_{20}f_{20}$	$x_{21}f_{21}$	$x_{22}f_{22}$	x_{23}	x_{24}
x_{30}	x_{31}	x_{32}	x_{33}	x_{34}
x_{40}	x_{41}	x_{42}	x_{43}	x_{44}

y_{00}	y_{01}
y_{10}	y_{11}

$$y_{00} = x_{00}f_{00} + x_{01}f_{01} + x_{02}f_{02} + x_{10}f_{10} + x_{11}f_{11} + x_{12}f_{12} + x_{20}f_{20} + x_{21}f_{21} + x_{22}f_{22}$$

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Forward Pass for stride > 1

- Forward Pass Equations

$$y_{00} = x_{00}f_{00} + x_{01}f_{01} + x_{02}f_{02} + x_{10}f_{10} + x_{11}f_{11} + x_{12}f_{12} + x_{20}f_{20} + x_{21}f_{21} + x_{22}f_{22}$$

$$y_{01} = x_{02}f_{00} + x_{03}f_{01} + x_{04}f_{02} + x_{12}f_{10} + x_{13}f_{11} + x_{14}f_{12} + x_{22}f_{20} + x_{23}f_{21} + x_{24}f_{22}$$

$$y_{10} = x_{20}f_{00} + x_{21}f_{01} + x_{22}f_{02} + x_{30}f_{10} + x_{31}f_{11} + x_{32}f_{12} + x_{40}f_{20} + x_{41}f_{21} + x_{42}f_{22}$$

$$y_{11} = x_{22}f_{00} + x_{23}f_{01} + x_{24}f_{02} + x_{32}f_{10} + x_{33}f_{11} + x_{34}f_{12} + x_{42}f_{20} + x_{43}f_{21} + x_{44}f_{22}$$

Derivative of Loss w.r.t. Filter

- Given the output derivative map we have to find the derivative of Loss w.r.t every Filter element

$\frac{\partial L}{\partial f_{00}}$	$\frac{\partial L}{\partial f_{01}}$	$\frac{\partial L}{\partial f_{02}}$
$\frac{\partial L}{\partial f_{10}}$	$\frac{\partial L}{\partial f_{11}}$	$\frac{\partial L}{\partial f_{12}}$
$\frac{\partial L}{\partial f_{20}}$	$\frac{\partial L}{\partial f_{21}}$	$\frac{\partial L}{\partial f_{22}}$

$\frac{\partial L}{\partial y_{00}}$	$\frac{\partial L}{\partial y_{01}}$
$\frac{\partial L}{\partial y_{10}}$	$\frac{\partial L}{\partial y_{11}}$

$$\frac{\partial L}{\partial f_{mn}} = \sum_{ij} \frac{\partial L}{\partial y_{ij}} \frac{\partial y_{ij}}{\partial f_{mn}}$$

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Derivative of Loss w.r.t. Filter

- Derivative of Loss w.r.t. *f00*

$x_{00}f_{00}$	$x_{01}f_{01}$	$x_{02}f_{02}$	x_{03}	x_{04}
$x_{10}f_{10}$	$x_{11}f_{11}$	$x_{12}f_{12}$	x_{13}	x_{14}
$x_{20}f_{20}$	$x_{21}f_{21}$	$x_{22}f_{22}$	x_{23}	x_{24}
x_{30}	x_{31}	x_{32}	x_{33}	x_{34}
x_{40}	x_{41}	x_{42}	x_{43}	x_{44}

x_{00}	x_{01}	$x_{02}f_{00}$	$x_{03}f_{01}$	$x_{04}f_{02}$
x_{20}	x_{21}	$x_{12}f_{10}$	$x_{13}f_{11}$	$x_{14}f_{12}$
x_{30}	x_{31}	$x_{22}f_{20}$	$x_{23}f_{21}$	$x_{24}f_{22}$
x_{30}	x_{31}	x_{32}	x_{33}	x_{34}
x_{40}	x_{41}	x_{42}	x_{43}	x_{44}

x_{00}	x_{01}	x_{02}	x_{03}	x_{04}
x_{20}	x_{21}	x_{22}	x_{13}	x_{14}
$x_{20}f_{00}$	$x_{21}f_{01}$	$x_{22}f_{02}$	x_{23}	x_{24}
$x_{30}f_{10}$	$x_{31}f_{11}$	$x_{32}f_{12}$	x_{33}	x_{34}
$x_{40}f_{20}$	$x_{41}f_{21}$	$x_{42}f_{22}$	x_{43}	x_{44}

x_{00}	x_{01}	x_{02}	x_{03}	x_{04}
x_{20}	x_{21}	x_{22}	x_{13}	x_{14}
x_{30}	x_{31}	$x_{22}f_{00}$	$x_{23}f_{01}$	$x_{24}f_{02}$
x_{30}	x_{31}	$x_{32}f_{10}$	$x_{33}f_{11}$	$x_{34}f_{12}$
x_{40}	x_{41}	$x_{42}f_{20}$	$x_{43}f_{21}$	$x_{44}f_{22}$

y_{00}	y_{01}
y_{10}	y_{11}

First, consider f_{00}. It contributes to all outputs: y_{00} , y_{01}, y_{10}, and y_{11}

$$y_{00} = x_{00}f_{00} + x_{01}f_{01} + x_{02}f_{02} + x_{10}f_{10} + x_{11}f_{11} + x_{12}f_{12} + x_{20}f_{20} + x_{21}f_{20} + x_{22}f_{22}$$
$$y_{01} = x_{02}f_{00} + x_{03}f_{01} + x_{04}f_{02} + x_{12}f_{10} + x_{13}f_{11} + x_{14}f_{12} + x_{22}f_{20} + x_{23}f_{21} + x_{24}f_{22}$$
$$y_{10} = x_{20}f_{00} + x_{21}f_{01} + x_{22}f_{02} + x_{30}f_{10} + x_{31}f_{11} + x_{32}f_{12} + x_{40}f_{20} + x_{41}f_{20} + x_{42}f_{22}$$
$$y_{11} = x_{22}f_{00} + x_{23}f_{01} + x_{24}f_{02} + x_{32}f_{10} + x_{33}f_{11} + x_{34}f_{12} + x_{42}f_{20} + x_{43}f_{20} + x_{44}f_{22}$$

$$\frac{\partial \mathrm{L}}{\partial f_{mn}} = \sum_{ij} \frac{\partial L}{\partial y_{ij}} \frac{\partial y_{ij}}{\partial f_{mn}}. \text{ Thus, } \frac{\partial \mathrm{L}}{\partial f_{00}} = \frac{\partial L}{\partial y_{00}} x_{00} + \frac{\partial L}{\partial y_{01}} x_{02} + \frac{\partial L}{\partial y_{10}} x_{20} + \frac{\partial L}{\partial y_{11}} x_{22}$$

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Derivative of Loss w.r.t. Filter

- Derivative of Loss w.r.t. *f01*

$x_{00}f_{00}$	$x_{01}f_{01}$	$x_{02}f_{02}$	x_{03}	x_{04}
$x_{10}f_{10}$	$x_{11}f_{11}$	$x_{12}f_{12}$	x_{13}	x_{14}
$x_{20}f_{20}$	$x_{21}f_{21}$	$x_{22}f_{22}$	x_{23}	x_{24}
x_{30}	x_{31}	x_{32}	x_{33}	x_{34}
x_{40}	x_{41}	x_{42}	x_{43}	x_{44}

x_{00}	x_{01}	$x_{02}f_{00}$	$x_{03}f_{01}$	$x_{04}f_{02}$
x_{20}	x_{21}	$x_{12}f_{10}$	$x_{13}f_{11}$	$x_{14}f_{12}$
x_{30}	x_{31}	$x_{22}f_{20}$	$x_{23}f_{21}$	$x_{24}f_{22}$
x_{30}	x_{31}	x_{32}	x_{33}	x_{34}
x_{40}	x_{41}	x_{42}	x_{43}	x_{44}

x_{00}	x_{01}	x_{02}	x_{03}	x_{04}
x_{20}	x_{21}	x_{22}	x_{13}	x_{14}
$x_{20}f_{00}$	$x_{21}f_{01}$	$x_{22}f_{02}$	x_{23}	x_{24}
$x_{30}f_{10}$	$x_{31}f_{11}$	$x_{32}f_{12}$	x_{33}	x_{34}
$x_{40}f_{20}$	$x_{41}f_{21}$	$x_{42}f_{22}$	x_{43}	x_{44}

x_{00}	x_{01}	x_{02}	x_{03}	x_{04}
x_{20}	x_{21}	x_{22}	x_{13}	x_{14}
x_{30}	x_{31}	$x_{22}f_{00}$	$x_{23}f_{01}$	$x_{24}f_{02}$
x_{30}	x_{31}	$x_{32}f_{10}$	$x_{33}f_{11}$	$x_{34}f_{12}$
x_{40}	x_{41}	$x_{42}f_{20}$	$x_{43}f_{21}$	$x_{44}f_{22}$

y_{00}	y_{01}
y_{10}	y_{11}

Next, consider f_{01}. It contributes to all outputs: y_{00} , y_{01}, y_{10}, and y_{11}

$$
\begin{aligned}
y_{00} &= x_{00}f_{00} + x_{01}f_{01} + x_{02}f_{02} + x_{10}f_{10} + x_{11}f_{11} + x_{12}f_{12} + x_{20}f_{20} + x_{21}f_{20} + x_{22}f_{22} \\
y_{01} &= x_{02}f_{00} + x_{03}f_{01} + x_{04}f_{02} + x_{12}f_{10} + x_{13}f_{11} + x_{14}f_{12} + x_{22}f_{20} + x_{23}f_{21} + x_{24}f_{22} \\
y_{10} &= x_{20}f_{00} + x_{21}f_{01} + x_{22}f_{02} + x_{30}f_{10} + x_{31}f_{11} + x_{32}f_{12} + x_{40}f_{20} + x_{41}f_{20} + x_{42}f_{22} \\
y_{11} &= x_{22}f_{00} + x_{23}f_{01} + x_{24}f_{02} + x_{32}f_{10} + x_{33}f_{11} + x_{34}f_{12} + x_{42}f_{20} + x_{43}f_{20} + x_{44}f_{22}
\end{aligned}
$$

$$\frac{\partial \mathrm{L}}{\partial f_{mn}} = \sum_{ij} \frac{\partial L}{\partial y_{ij}} \frac{\partial y_{ij}}{\partial f_{mn}}. \text{ Thus, } \frac{\partial \mathrm{L}}{\partial f_{01}} = \frac{\partial L}{\partial y_{00}} x_{01} + \frac{\partial L}{\partial y_{01}} x_{03} + \frac{\partial L}{\partial y_{10}} x_{21} + \frac{\partial L}{\partial y_{11}} x_{23}$$

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Derivative of Loss w.r.t. Filter

- Derivative of Loss w.r.t. *f11*

$x_{00}f_{00}$	$x_{01}f_{01}$	$x_{02}f_{02}$	x_{03}	x_{04}
$x_{10}f_{10}$	$x_{11}f_{11}$	$x_{12}f_{12}$	x_{13}	x_{14}
$x_{20}f_{20}$	$x_{21}f_{21}$	$x_{22}f_{22}$	x_{23}	x_{24}
x_{30}	x_{31}	x_{32}	x_{33}	x_{34}
x_{40}	x_{41}	x_{42}	x_{43}	x_{44}

x_{00}	x_{01}	$x_{02}f_{00}$	$x_{03}f_{01}$	$x_{04}f_{02}$
x_{20}	x_{21}	$x_{12}f_{10}$	$x_{13}f_{11}$	$x_{14}f_{12}$
x_{30}	x_{31}	$x_{22}f_{20}$	$x_{23}f_{21}$	$x_{24}f_{22}$
x_{30}	x_{31}	x_{32}	x_{33}	x_{34}
x_{40}	x_{41}	x_{42}	x_{43}	x_{44}

x_{00}	x_{01}	x_{02}	x_{03}	x_{04}
x_{20}	x_{21}	x_{22}	x_{13}	x_{14}
$x_{20}f_{00}$	$x_{21}f_{01}$	$x_{22}f_{02}$	x_{23}	x_{24}
$x_{30}f_{10}$	$x_{31}f_{11}$	$x_{32}f_{12}$	x_{33}	x_{34}
$x_{40}f_{20}$	$x_{41}f_{21}$	$x_{42}f_{22}$	x_{43}	x_{44}

x_{00}	x_{01}	x_{02}	x_{03}	x_{04}
x_{20}	x_{21}	x_{22}	x_{13}	x_{14}
x_{30}	x_{31}	$x_{22}f_{00}$	$x_{23}f_{01}$	$x_{24}f_{02}$
x_{30}	x_{31}	$x_{32}f_{10}$	$x_{33}f_{11}$	$x_{34}f_{12}$
x_{40}	x_{41}	$x_{42}f_{20}$	$x_{43}f_{21}$	$x_{44}f_{22}$

y_{00}	y_{01}
y_{10}	y_{11}

Next, consider f_{11}. It contributes to all outputs: y_{00} , y_{01}, y_{10}, and y_{11}

$$
\begin{aligned}
y_{00} &= x_{00}f_{00} + x_{01}f_{01} + x_{02}f_{02} + x_{10}f_{10} + x_{11}f_{11} + x_{12}f_{12} + x_{20}f_{20} + x_{21}f_{20} + x_{22}f_{22} \\
y_{01} &= x_{02}f_{00} + x_{03}f_{01} + x_{04}f_{02} + x_{12}f_{10} + x_{13}f_{11} + x_{14}f_{12} + x_{22}f_{20} + x_{23}f_{21} + x_{24}f_{22} \\
y_{10} &= x_{20}f_{00} + x_{21}f_{01} + x_{22}f_{02} + x_{30}f_{10} + x_{31}f_{11} + x_{32}f_{12} + x_{40}f_{20} + x_{41}f_{20} + x_{42}f_{22} \\
y_{11} &= x_{22}f_{00} + x_{23}f_{01} + x_{24}f_{02} + x_{32}f_{10} + x_{33}f_{11} + x_{34}f_{12} + x_{42}f_{20} + x_{43}f_{20} + x_{44}f_{22}
\end{aligned}
$$

$$\frac{\partial \mathrm{L}}{\partial f_{mn}} = \sum_{ij} \frac{\partial L}{\partial y_{ij}} \frac{\partial y_{ij}}{\partial f_{mn}}. \text{ Thus, } \frac{\partial \mathrm{L}}{\partial f_{11}} = \frac{\partial L}{\partial y_{00}} x_{11} + \frac{\partial L}{\partial y_{01}} x_{13} + \frac{\partial L}{\partial y_{10}} x_{31} + \frac{\partial L}{\partial y_{11}} x_{33}$$

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Derivative of Loss w.r.t. Filter

- Derivative of Loss w.r.t. *f22*

$x_{00}f_{00}$	$x_{01}f_{01}$	$x_{02}f_{02}$	x_{03}	x_{04}
$x_{10}f_{10}$	$x_{11}f_{11}$	$x_{12}f_{12}$	x_{13}	x_{14}
$x_{20}f_{20}$	$x_{21}f_{21}$	$x_{22}f_{22}$	x_{23}	x_{24}
x_{30}	x_{31}	x_{32}	x_{33}	x_{34}
x_{40}	x_{41}	x_{42}	x_{43}	x_{44}

x_{00}	x_{01}	$x_{02}f_{00}$	$x_{03}f_{01}$	$x_{04}f_{02}$
x_{20}	x_{21}	$x_{12}f_{10}$	$x_{13}f_{11}$	$x_{14}f_{12}$
x_{30}	x_{31}	$x_{22}f_{20}$	$x_{23}f_{21}$	$x_{24}f_{22}$
x_{30}	x_{31}	x_{32}	x_{33}	x_{34}
x_{40}	x_{41}	x_{42}	x_{43}	x_{44}

x_{00}	x_{01}	x_{02}	x_{03}	x_{04}
x_{20}	x_{21}	x_{22}	x_{13}	x_{14}
$x_{20}f_{00}$	$x_{21}f_{01}$	$x_{22}f_{02}$	x_{23}	x_{24}
$x_{30}f_{10}$	$x_{31}f_{11}$	$x_{32}f_{12}$	x_{33}	x_{34}
$x_{40}f_{20}$	$x_{41}f_{21}$	$x_{42}f_{22}$	x_{43}	x_{44}

x_{00}	x_{01}	x_{02}	x_{03}	x_{04}
x_{20}	x_{21}	x_{22}	x_{13}	x_{14}
x_{30}	x_{31}	$x_{22}f_{00}$	$x_{23}f_{01}$	$x_{24}f_{02}$
x_{30}	x_{31}	$x_{32}f_{10}$	$x_{33}f_{11}$	$x_{34}f_{12}$
x_{40}	x_{41}	$x_{42}f_{20}$	$x_{43}f_{21}$	$x_{44}f_{22}$

y_{00}	y_{01}
y_{10}	y_{11}

Finally, consider f_{22}. It contributes to all outputs: y_{00} , y_{01}, y_{10}, and y_{11}

$$
\begin{aligned}
y_{00} &= x_{00}f_{00} + x_{01}f_{01} + x_{02}f_{02} + x_{10}f_{10} + x_{11}f_{11} + x_{12}f_{12} + x_{20}f_{20} + x_{21}f_{20} + x_{22}f_{22} \\
y_{01} &= x_{02}f_{00} + x_{03}f_{01} + x_{04}f_{02} + x_{12}f_{10} + x_{13}f_{11} + x_{14}f_{12} + x_{22}f_{20} + x_{23}f_{21} + x_{24}f_{22} \\
y_{10} &= x_{20}f_{00} + x_{21}f_{01} + x_{22}f_{02} + x_{30}f_{10} + x_{31}f_{11} + x_{32}f_{12} + x_{40}f_{20} + x_{41}f_{20} + x_{42}f_{22} \\
y_{11} &= x_{22}f_{00} + x_{23}f_{01} + x_{24}f_{02} + x_{32}f_{10} + x_{33}f_{11} + x_{34}f_{12} + x_{42}f_{20} + x_{43}f_{20} + x_{44}f_{22}
\end{aligned}
$$

$$\frac{\partial \mathrm{L}}{\partial f_{mn}} = \sum_{ij} \frac{\partial L}{\partial y_{ij}} \frac{\partial y_{ij}}{\partial f_{mn}}. \text{ Thus, } \frac{\partial \mathrm{L}}{\partial f_{22}} = \frac{\partial L}{\partial y_{00}} x_{22} + \frac{\partial L}{\partial y_{01}} x_{24} + \frac{\partial L}{\partial y_{10}} x_{42} + \frac{\partial L}{\partial y_{11}} x_{44}$$

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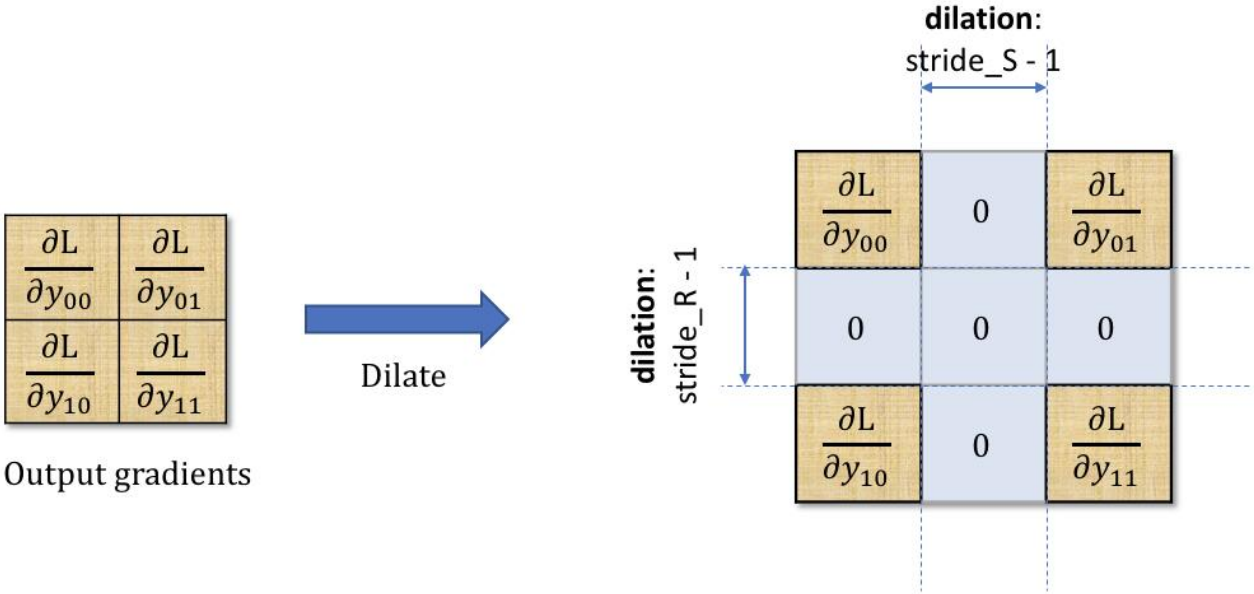
Derivative of Loss w.r.t. Filter

- To view the operation as a convolution operation we need to modify the output derivative map as follows:
  - **Dilate/Upsample with stride-1 zeros**



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Derivative of Loss w.r.t. Filter

Convolution(Input, Upsampled output derivative map)

$$\frac{\partial \mathrm{L}}{\partial f_{00}} = \frac{\partial L}{\partial y_{00}} x_{00} + \frac{\partial L}{\partial y_{01}} x_{02} + \frac{\partial L}{\partial y_{10}} x_{20} + \frac{\partial L}{\partial y_{11}} x_{22}$$

$\frac{\partial \mathrm{L}}{\partial f_{00}}$	$\frac{\partial \mathrm{L}}{\partial f_{01}}$	$\frac{\partial \mathrm{L}}{\partial f_{02}}$
$\frac{\partial \mathrm{L}}{\partial f_{10}}$	$\frac{\partial \mathrm{L}}{\partial f_{11}}$	$\frac{\partial \mathrm{L}}{\partial f_{12}}$
$\frac{\partial \mathrm{L}}{\partial f_{20}}$	$\frac{\partial \mathrm{L}}{\partial f_{21}}$	$\frac{\partial \mathrm{L}}{\partial f_{22}}$

=

$\frac{\partial \mathrm{L}}{\partial y_{00}} x_{00}$	$0 * x_{01}$	$\frac{\partial \mathrm{L}}{\partial y_{01}} x_{02}$	x_{03}	x_{04}
$0 * x_{10}$	$0 * x_{11}$	$0 * x_{12}$	x_{13}	x_{14}
$\frac{\partial \mathrm{L}}{\partial y_{10}} x_{20}$	$0 * x_{21}$	$\frac{\partial \mathrm{L}}{\partial y_{11}} x_{22}$	x_{23}	x_{24}
x_{30}	x_{31}	x_{32}	x_{33}	x_{34}
x_{40}	x_{41}	x_{42}	x_{43}	x_{44}

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Derivative of Loss w.r.t. Input

- We have to find Derivative of Loss w.r.t to every input which is given as follows:

$$\frac{\partial \mathrm{L}}{\partial x_{mn}} = \sum_{ij} \frac{\partial L}{\partial y_{ij}} \frac{\partial y_{ij}}{\partial x_{mn}}$$

Derivative of Loss w.r.t. Input

- We have to find Derivative of Loss w.r.t *x22*

$$\frac{\partial \mathrm{L}}{\partial x_{mn}} = \sum_{ij} \frac{\partial L}{\partial y_{ij}} \frac{\partial y_{ij}}{\partial x_{mn}}$$

x_{00}	x_{01}	x_{02}	x_{03}	x_{04}
x_{10}	x_{11}	x_{12}	x_{13}	x_{14}
x_{20}	x_{21}	x_{22}	x_{23}	x_{24}
x_{30}	x_{31}	x_{32}	x_{33}	x_{34}
x_{40}	x_{41}	x_{42}	x_{43}	x_{44}

y_{00}	y_{01}
y_{10}	y_{11}

Finally, consider x_{22}. It contributes to all outputs: y_{00} , y_{01}, y_{10}, and y_{11}

$$y_{00} = x_{00}f_{00} + x_{01}f_{01} + x_{02}f_{02} + x_{10}f_{10} + x_{11}f_{11} + x_{12}f_{12} + x_{20}f_{20} + x_{21}f_{20} + x_{22}f_{22}$$
$$y_{01} = x_{02}f_{00} + x_{03}f_{01} + x_{04}f_{02} + x_{12}f_{10} + x_{13}f_{11} + x_{14}f_{12} + x_{22}f_{20} + x_{23}f_{21} + x_{24}f_{22}$$
$$y_{10} = x_{20}f_{00} + x_{21}f_{01} + x_{22}f_{02} + x_{30}f_{10} + x_{31}f_{11} + x_{32}f_{12} + x_{40}f_{20} + x_{41}f_{20} + x_{42}f_{22}$$
$$y_{11} = x_{22}f_{00} + x_{23}f_{01} + x_{24}f_{02} + x_{32}f_{10} + x_{33}f_{11} + x_{34}f_{12} + x_{42}f_{20} + x_{43}f_{20} + x_{44}f_{22}$$

$$\text{Thus, } \frac{\partial \mathrm{L}}{\partial x_{22}} = \frac{\partial L}{\partial y_{00}} f_{22} + \frac{\partial L}{\partial y_{01}} f_{20} + \frac{\partial L}{\partial y_{10}} f_{20} + \frac{\partial L}{\partial y_{11}} f_{00}$$

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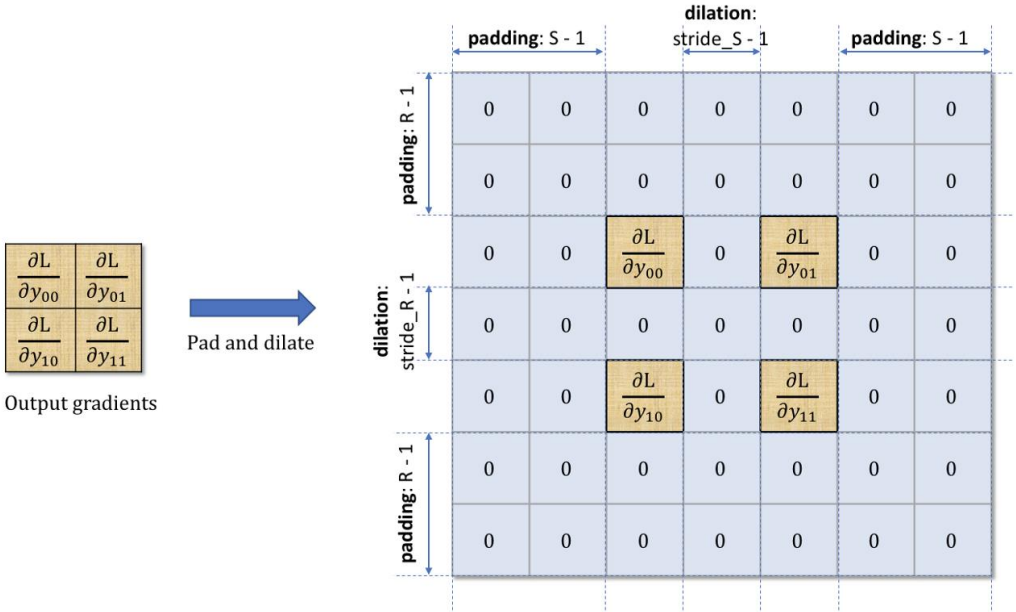
Convolution in Backprop

- In order to view the previous operation as convolution, we will have to do following modifications -
  - **Dilate/Upsample output derivative map with stride - 1**
  - **Pad the output derivative map with kernel_width - 1 & kernel_height - 1**



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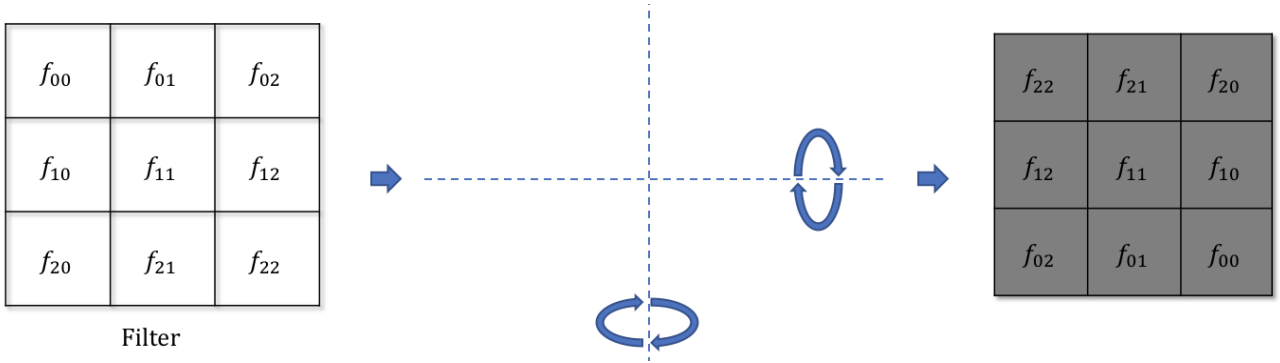
Convolution in Backprop

- In order to view the previous operation as convolution, we will have to do following modifications -
  - **Dilate/Upsample output derivative map with stride - 1**
  - **Pad output derivative map with kernel_width - 1 and kernel_height - 1**
  - **Flip the filter horizontally and vertically**



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Derivative of Loss w.r.t Input

Convolution (Flipped Filter, Padded and Upsampled Output Derivative Map)

$$\frac{\partial L}{\partial x_{00}} = \frac{\partial L}{\partial y_{00}} f_{00}$$

$\frac{\partial L}{\partial x_{00}}$	$\frac{\partial L}{\partial x_{01}}$	$\frac{\partial L}{\partial x_{02}}$	$\frac{\partial L}{\partial x_{03}}$	$\frac{\partial L}{\partial x_{04}}$
$\frac{\partial L}{\partial x_{10}}$	$\frac{\partial L}{\partial x_{11}}$	$\frac{\partial L}{\partial x_{12}}$	$\frac{\partial L}{\partial x_{13}}$	$\frac{\partial L}{\partial x_{14}}$
$\frac{\partial L}{\partial x_{20}}$	$\frac{\partial L}{\partial x_{21}}$	$\frac{\partial L}{\partial x_{22}}$	$\frac{\partial L}{\partial x_{23}}$	$\frac{\partial L}{\partial x_{24}}$
$\frac{\partial L}{\partial x_{30}}$	$\frac{\partial L}{\partial x_{31}}$	$\frac{\partial L}{\partial x_{32}}$	$\frac{\partial L}{\partial x_{33}}$	$\frac{\partial L}{\partial x_{34}}$
$\frac{\partial L}{\partial x_{40}}$	$\frac{\partial L}{\partial x_{41}}$	$\frac{\partial L}{\partial x_{42}}$	$\frac{\partial L}{\partial x_{43}}$	$\frac{\partial L}{\partial x_{44}}$

=

$0 * f_{22}$	$0 * f_{21}$	$0 * f_{20}$	0	0	0	0
$0 * f_{12}$	$0 * f_{11}$	$0 * f_{10}$	0	0	0	0
$0 * f_{02}$	$0 * f_{01}$	$\frac{\partial L}{\partial y_{00}} f_{00}$	0	$\frac{\partial L}{\partial y_{01}}$	0	0
0	0	0	0	0	0	0
0	0	$\frac{\partial L}{\partial y_{10}}$	0	$\frac{\partial L}{\partial y_{11}}$	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0

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Takeaway - Stride > 1 BackProp

- Both the forward and backward pass of a convolution layer can be viewed as Convolution operations
- Derivative of Loss w.r.t filter

Convolution(Input, Upsampled output derivative map)

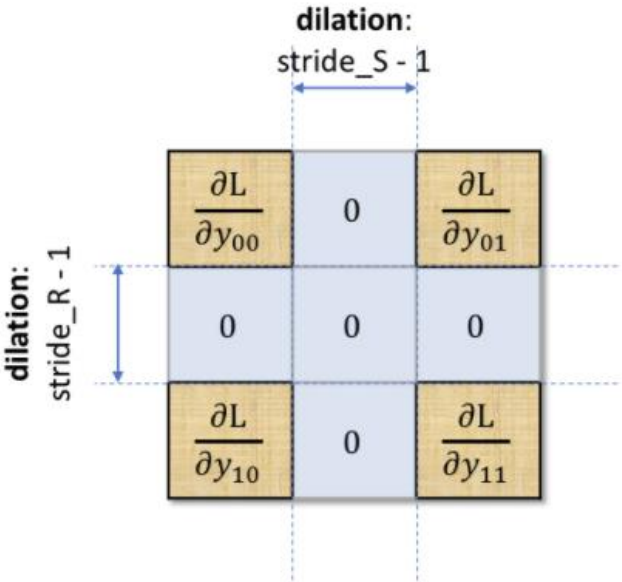
- Derivative of Loss w.r.t. Input

Convolution (Flipped Filter, Padded and Upsampled Output Derivative Map)

Takeaway - Stride › 1 BackProp

- Stride > 1 Backpropagation can also be viewed as

Upsample Output Derivative Map → Stride 1 Convolution Backward



$$\frac{\partial L}{\partial F} = \text{Convolution}\left(\text{Input } X,\ \text{Loss gradient } \frac{\partial L}{\partial O}\right)$$

$$\frac{\partial L}{\partial X} = \text{Full Convolution}\left(\text{180}^\circ\text{ rotated Filter } F,\ \text{Loss Gradient } \frac{\partial L}{\partial O}\right)$$

Here, $\frac{\partial L}{\partial y_{ij}} = \frac{\partial L}{\partial O}$

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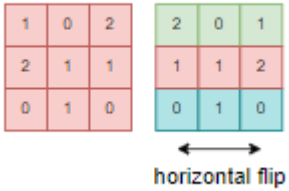
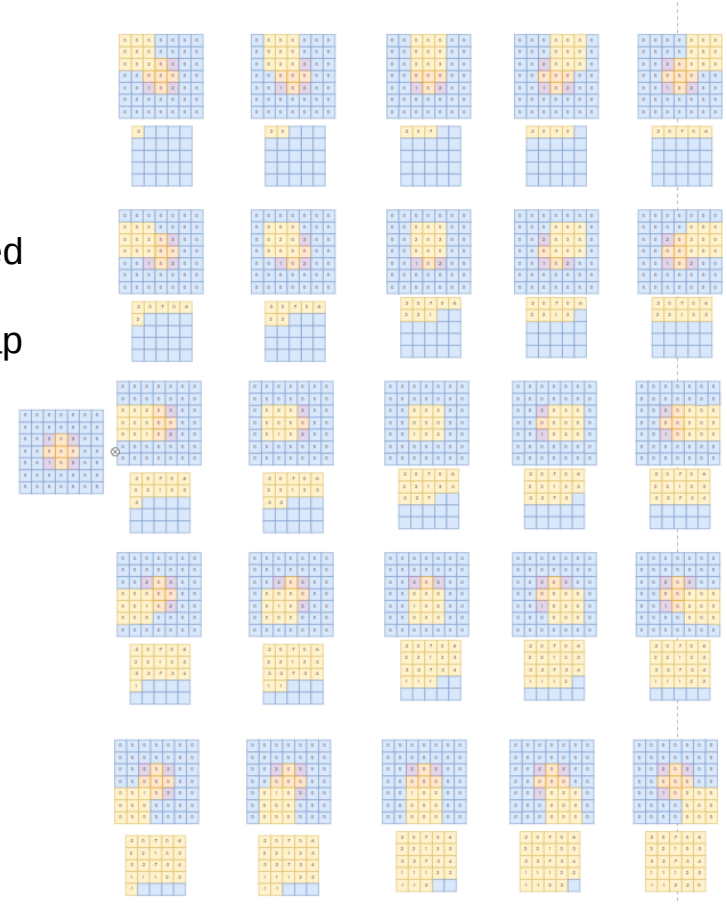
BackProp: Derivative of Loss w.r.t Input



Filter and it's flipped version

Convolve with Padded and upsampled Output Derivative Map



References

https://medium.com/@mayank.utexas/backpropagation-for-convolution-with-strides-8137e4fc2710

https://medium.com/@mayank.utexas/backpropagation-for-convolution-with-stridesfb2f2efc4faa

https://medium.com/@pavisj/convolutions-and-backpropagations-46026a8f5d2c

https://deeplearning.cs.cmu.edu/S22/document/recitation/Recitation5/Rec5.zip