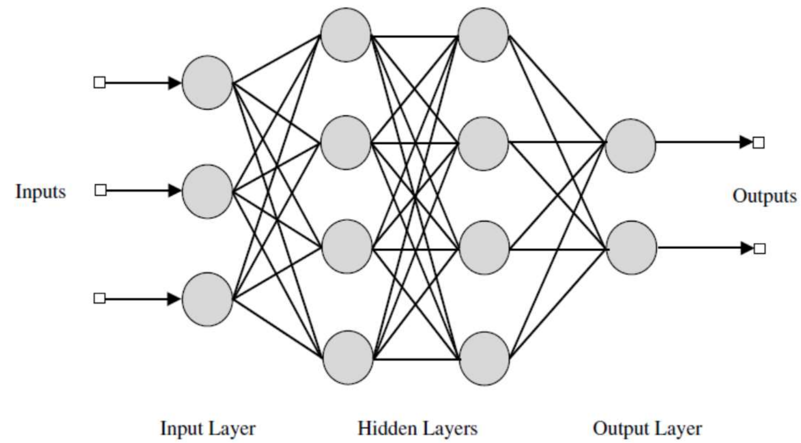
Neural Networks

Hopfield Nets and Auto Associators

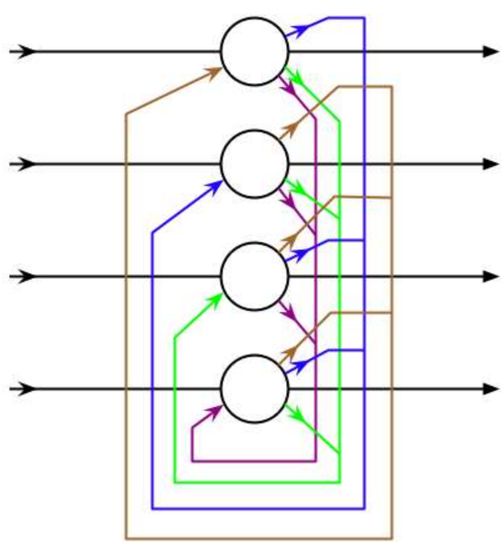
Fall 2022

Story so far

- **Neural networks for computation**
- All feedforward structures



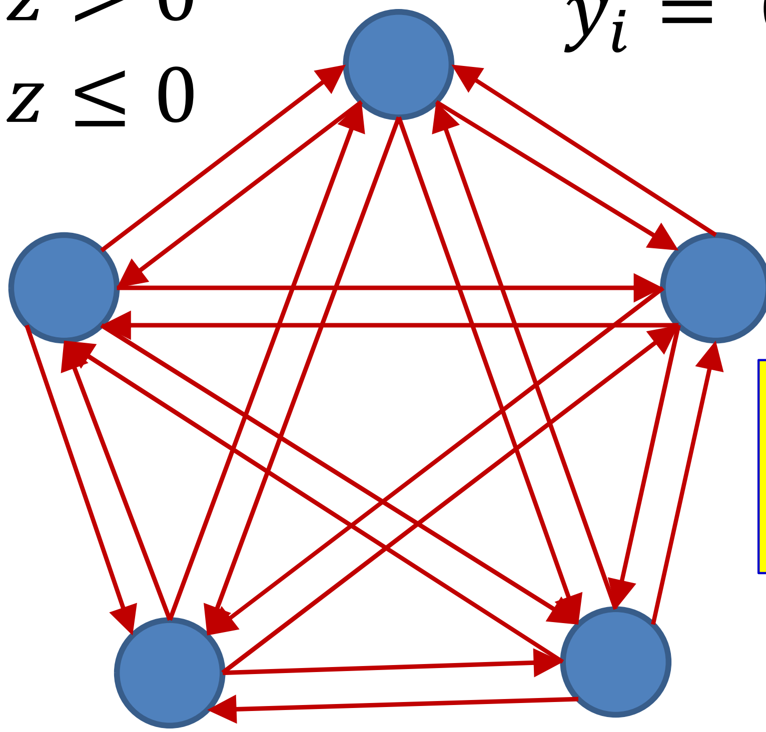
- But what about..



Consider this loopy network

$$\Theta(z) = \begin{cases} +1 \; if \; z > 0 \\ -1 \; if \; z \leq 0 \end{cases}$$

$$y_i = \Theta\left(\sum_{j \neq i} w_{ji} y_j + b_i\right)$$



The output of a neuron affects the input to the neuron

- Each neuron is a perceptron with +1/-1 output
- Every neuron *receives* input from every other neuron
- Every neuron *outputs* signals to every other neuron

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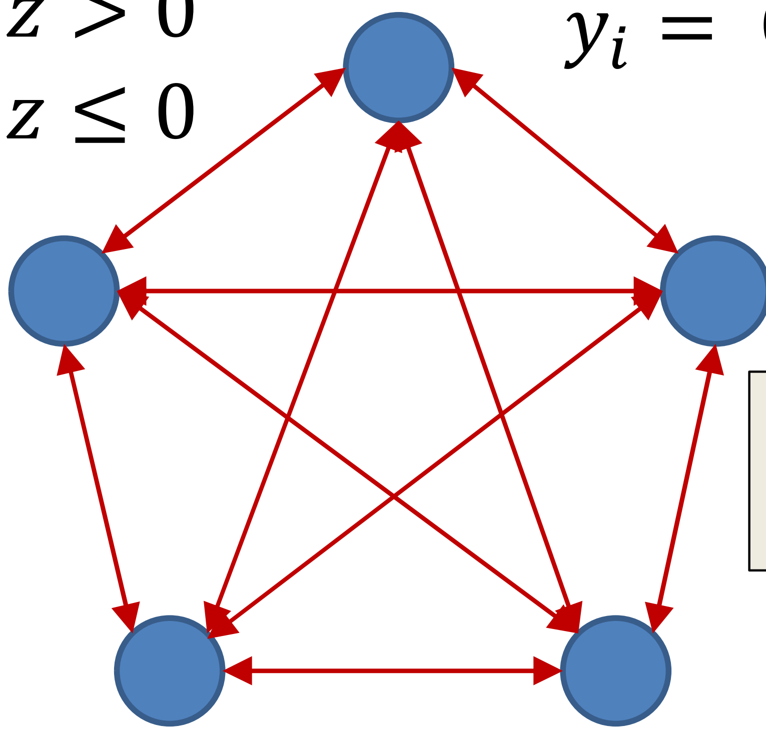

A symmetric network:
$w_{ij} = w_{ji}$

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Hopfield Net

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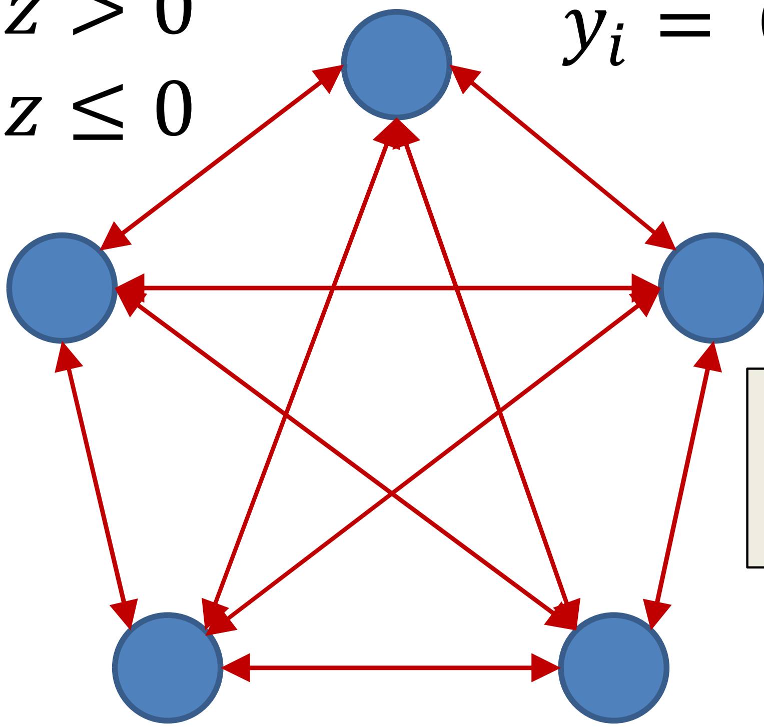

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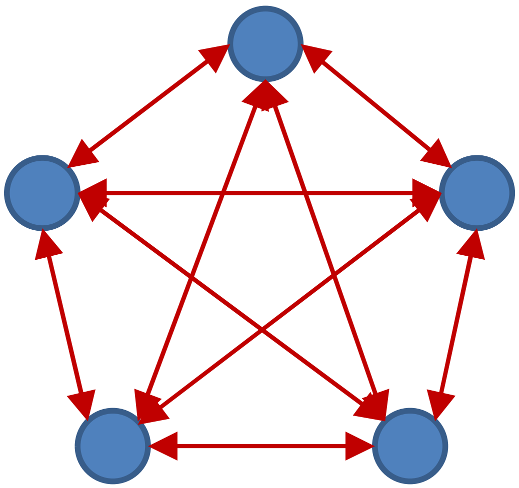
Loopy network



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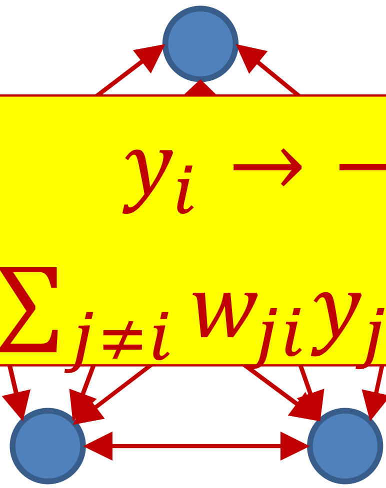
- At each time each neuron receives a "field" $\sum_{j \neq i} w_{ji} y_j + b_i$
- If the sign of the field matches its own sign, it does not respond
- If the sign of the field opposes its own sign, it "flips" to match the sign of the field

Loopy network



$$y_i \rightarrow -y_i$$

if $y_i\left(\sum_{j\neq i} w_{ji}y_j + b_i\right) < 0$

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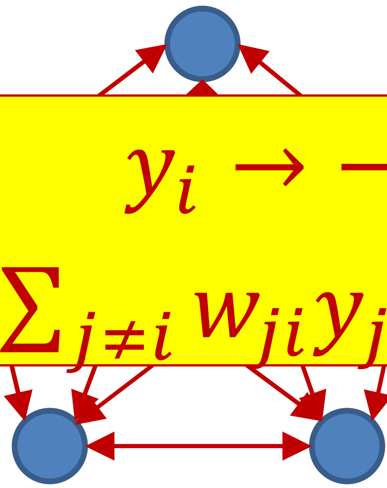

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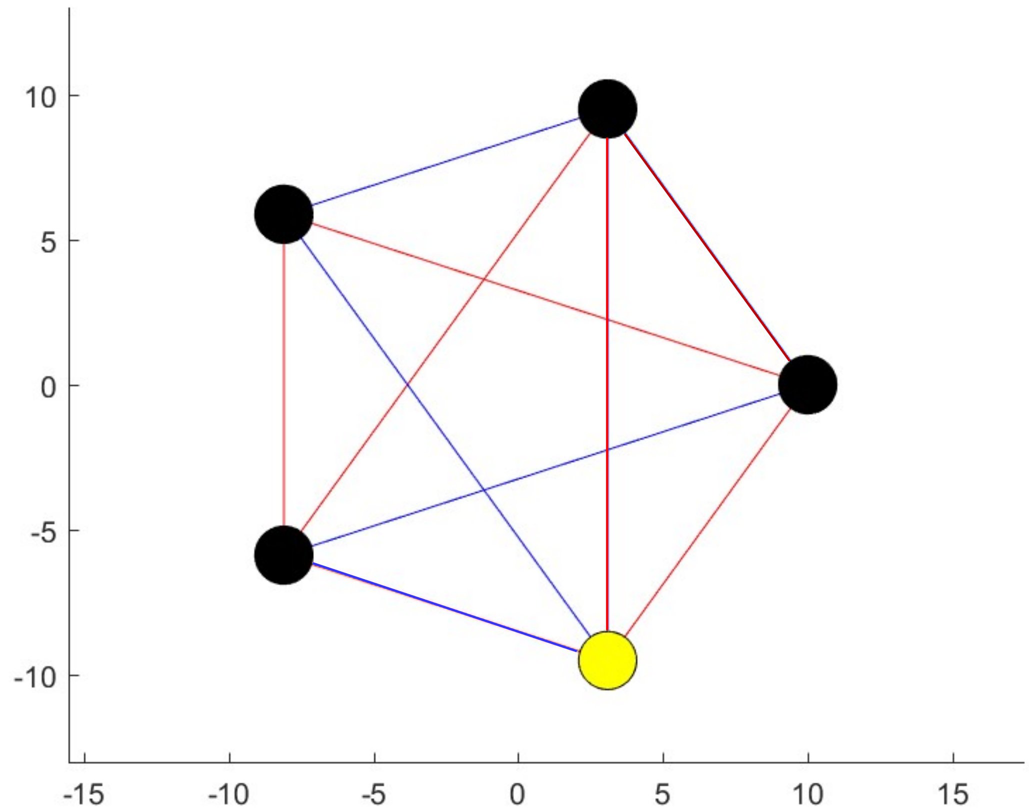
$$\Theta(z) = \begin{cases} +1 \; if \; z > 0 \\ -1 \; if \; z \leq 0 \end{cases}$$

A neuron "flips" if weighted sum of other neurons' outputs is of the opposite sign to its own current (output) value

But this may cause *other* neurons to flip!

- …es a "field" $\sum_{j\neq i} w_{ji}y_j + b_i$
- …s own sign, it does not respond
- If the sign of the field opposes its own sign, it "flips" to match the sign of the field

Example



- Red edges are +1, blue edges are -1
- Yellow nodes are -1, black nodes are +1

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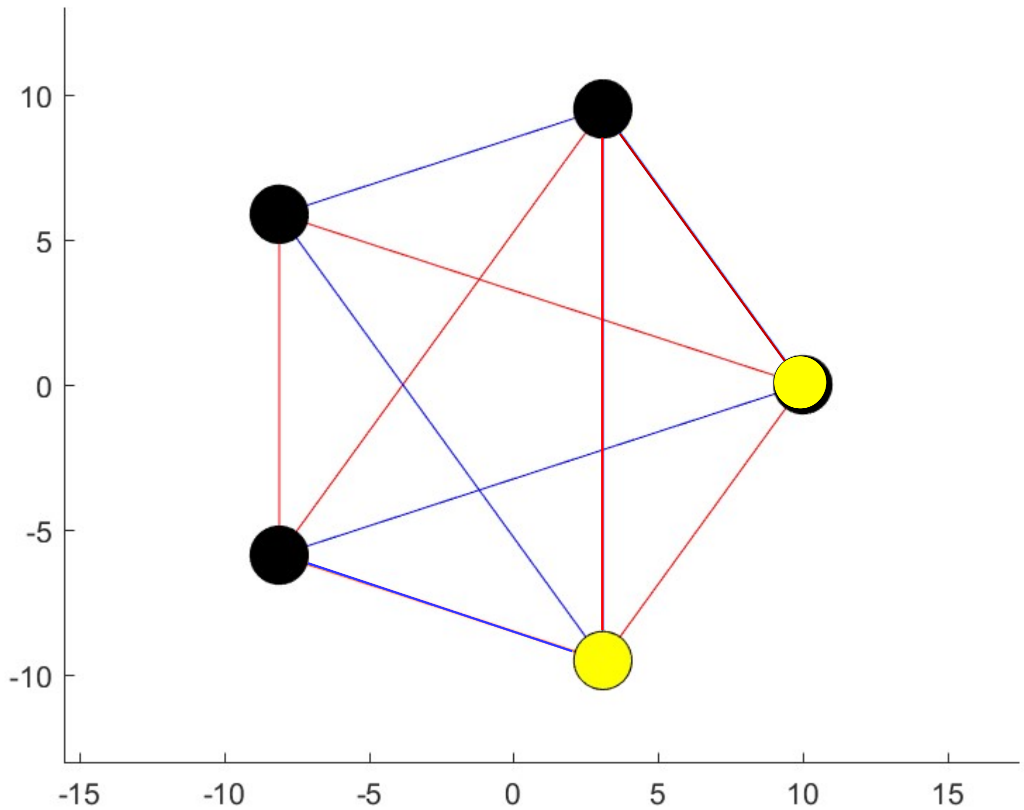

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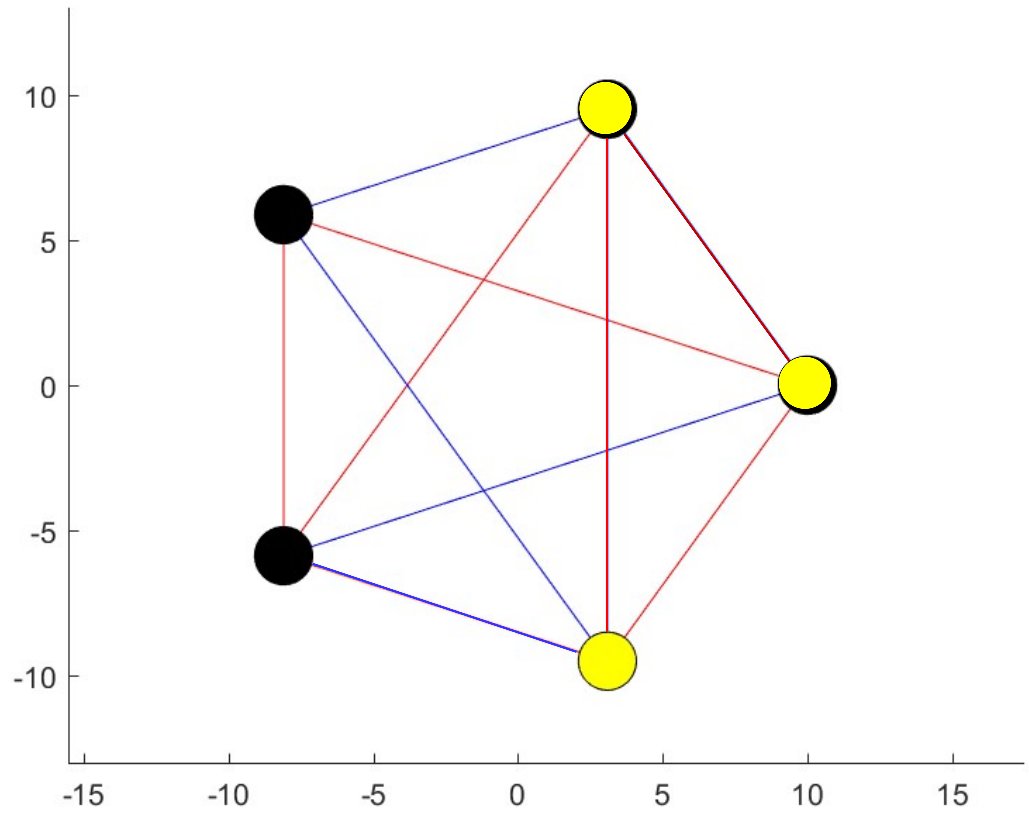

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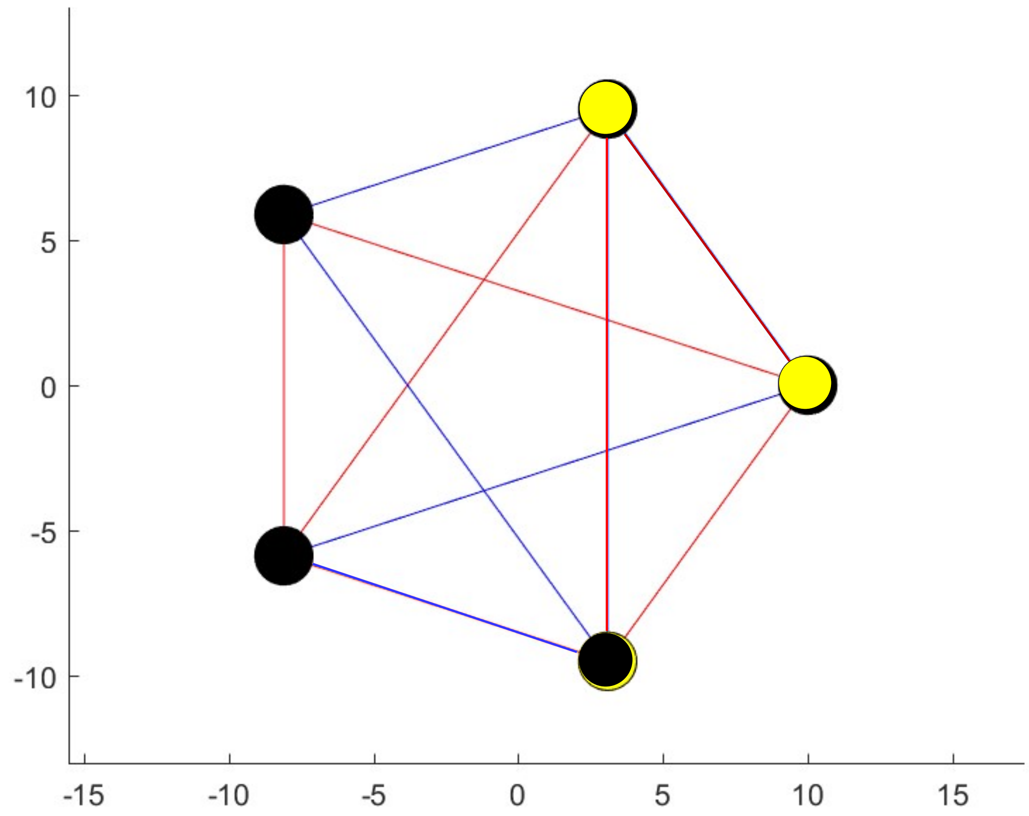

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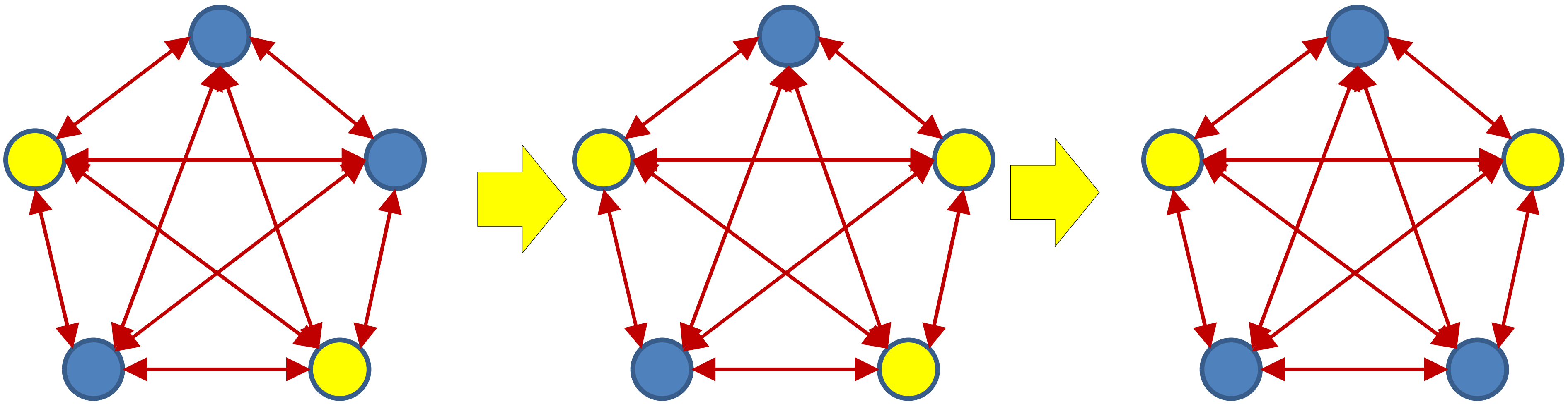
Loopy network

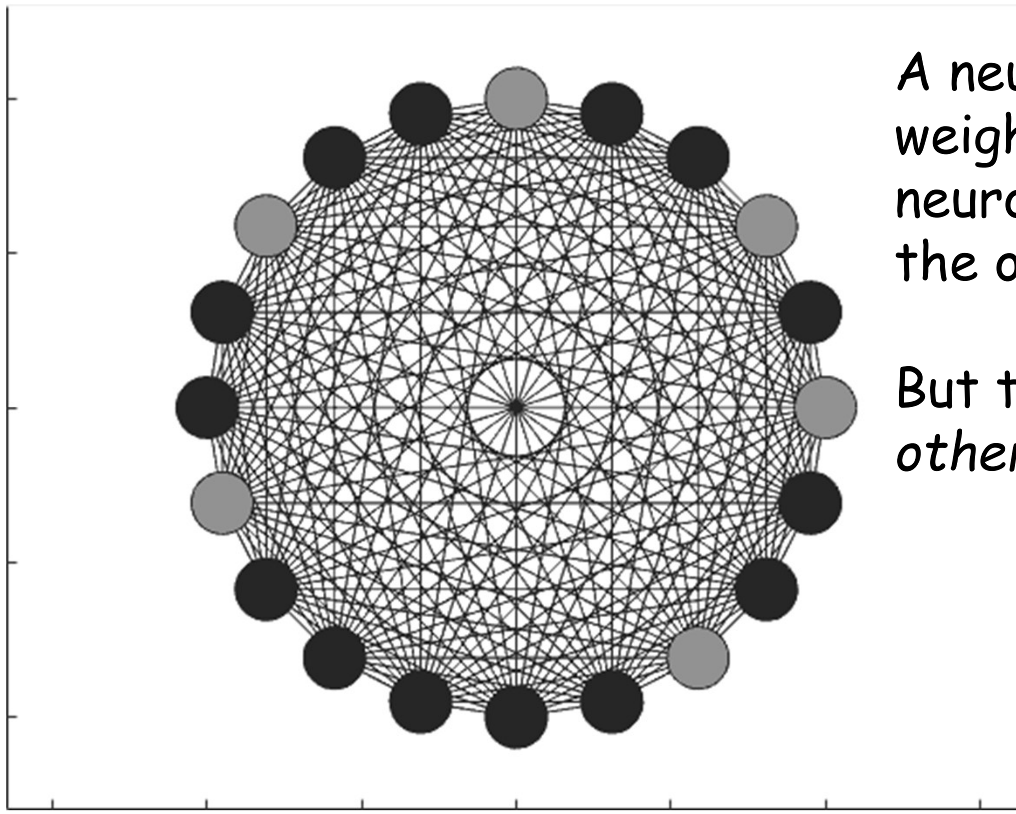


- If the sign of the field at any neuron opposes its own sign, it “flips” to match the field
  - Which will change the field at other nodes
    - Which may then flip
      - Which may cause other neurons including the first one to flip…
        - » And so on…

20 evolutions of a loopy net

$$\Theta(z) = \begin{cases} +1 \ if \ z > 0 \\ -1 \ if \ z \leq 0 \end{cases}$$

$$y_i = \Theta\left(\sum_{j \neq i} w_{ji} y_j + b_i\right)$$

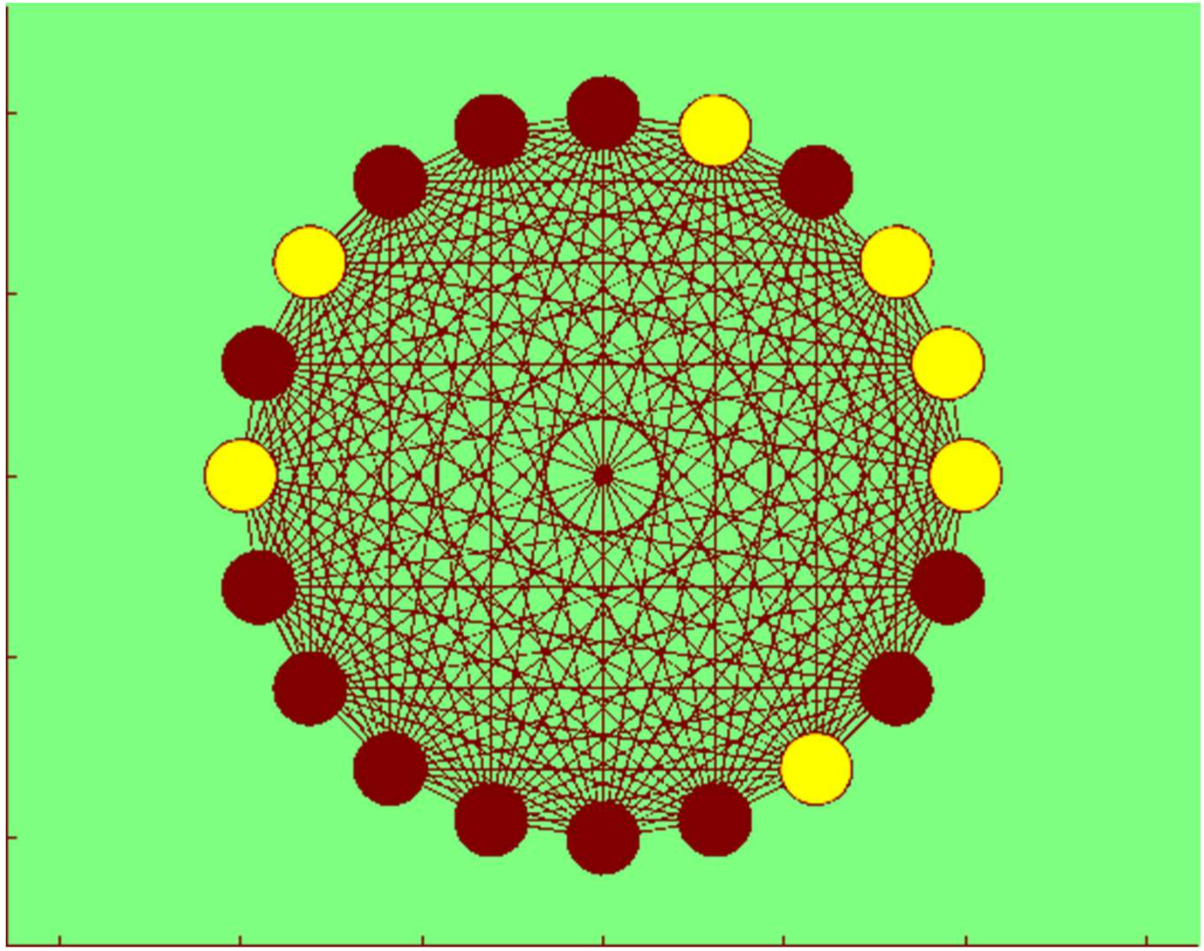


A neuron "flips" if weighted sum of other neuron's outputs is of the opposite sign

But this may cause *other* neurons to flip!

- All neurons which do not "align" with the local field "flip"

120 evolutions of a loopy net



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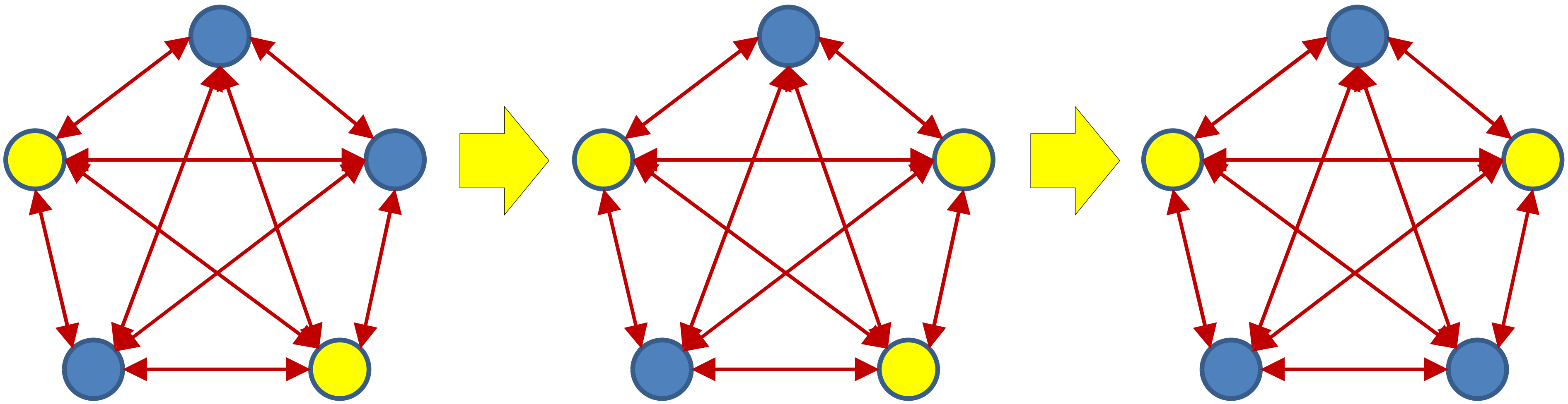
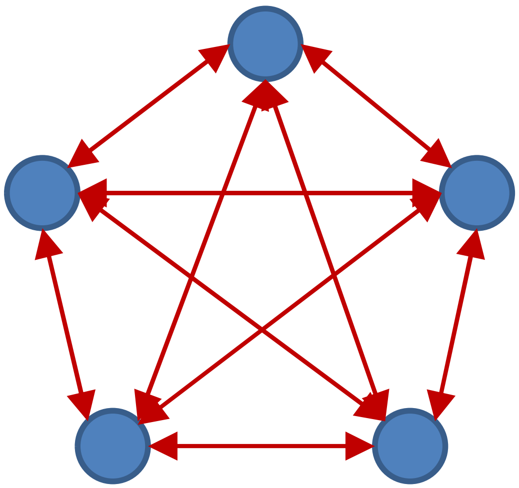
Loopy network



- If the sign of the field at any neuron opposes its own sign, it "flips" to match the field
  - Which will change the field at other nodes
    - Which may then flip
      - Which may cause other neurons including the first one to flip…

- *Will this behavior continue for ever??*

Loopy network



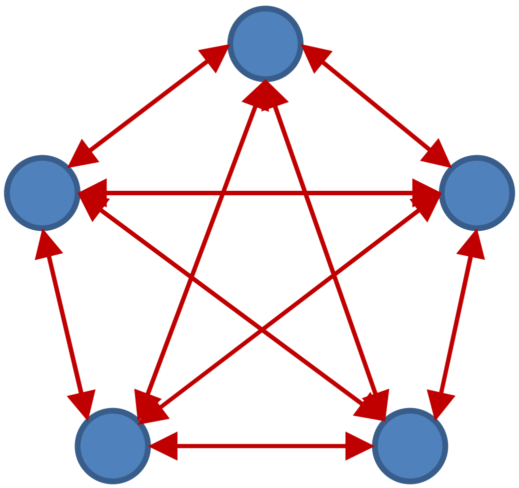
$$y_i = \Theta\left(\sum_{j \neq i} w_{ji} y_j + b_i\right)$$

$$\Theta(z) = \begin{cases} +1 \; if \; z > 0 \\ -1 \; if \; z \leq 0 \end{cases}$$

- Let y_i^- be the output of the *i*-th neuron just *before* it responds to the current field
- Let y_i^+ be the output of the *i*-th neuron just *after* it responds to the current field
- If $y_i^- = sign\left(\sum_{j \neq i} w_{ji} y_j + b_i\right)$, then $y_i^+ = y_i^-$
  - If the sign of the field matches its own sign, it does not flip

$$y_i^+\left(\sum_{j \neq i} w_{ji} y_j + b_i\right) - y_i^-\left(\sum_{j \neq i} w_{ji} y_j + b_i\right) = 0$$

Loopy network



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- If $y_i^- \neq sign\left(\sum_{j \neq i} w_{ji} y_j + b_i\right)$, then $y_i^+ = -y_i^-$

$$y_i^+\left(\sum_{j \neq i} w_{ji} y_j + b_i\right) - y_i^-\left(\sum_{j \neq i} w_{ji} y_j + b_i\right) = 2y_i^+\left(\sum_{j \neq i} w_{ji} y_j + b_i\right)$$

  – This term is always positive!

- *Every flip of a neuron is guaranteed to locally increase*

$$y_i\left(\sum_{j \neq i} w_{ji} y_j + b_i\right)$$

Globally

- Consider the following sum across *all* nodes

$$D(y_1, y_2, \dots, y_N) = \sum_i y_i \left(\sum_{j \neq i} w_{ji} y_j + b_i \right)$$

$$= \sum_{i,j \neq i} w_{ij} y_i y_j + \sum_i b_i y_i$$

  - Assume $w_{ii} = 0$

- For any unit k that "flips" because of the local field

$$\Delta D(y_k) = D\left(y_1, \dots, y_k^+, \dots, y_N\right) - D\left(y_1, \dots, y_k^-, \dots, y_N\right)$$

- This is strictly positive

$$\Delta D(y_k) = 2 y_k^+ \left(\sum_{j \neq k} w_{jk} y_j + b_k \right)$$

Upon flipping a single unit

$$\Delta D(y_k) = D(y_1, \dots, y_k^+, \dots, y_N) - D(y_1, \dots, y_k^-, \dots, y_N)$$

- Expanding

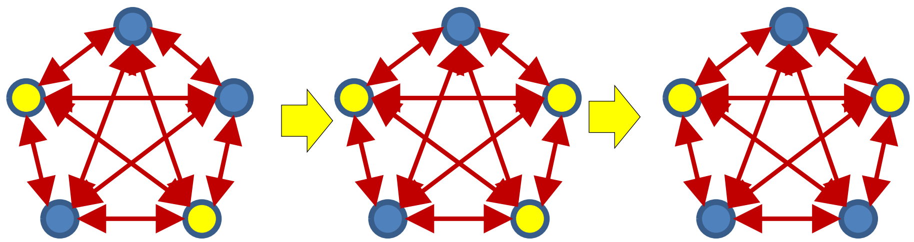
$$\Delta D(y_k) = (y_k^+ - y_k^-)\left(\sum_{j \neq k} w_{jk} y_j + b_k\right)$$

  - All other terms that do not include y_k cancel out

- This is always positive!

- *Every flip of a unit results in an increase in* D

Hopfield Net



- Flipping a unit will result in an increase (non-decrease) of

$$D = \sum_{i,j\neq i} w_{ij} y_i y_j + \sum_i b_i y_i$$

- D is bounded

$$D_{max} = \sum_{i,j\neq i} |w_{ij}| + \sum_i |b_i|$$

- The minimum increment of D in a flip is

$$\Delta D_{min} = \min_{i,\{y_i,\, i=1..N\}} 2 \left| \sum_{j\neq i} w_{ji} y_j + b_i \right|$$

- Any sequence of flips must converge in a finite number of steps

The Energy of a Hopfield Net

- Define the *Energy* of the network as

$$E = -\frac{1}{2}\left(\sum_{i,j\neq i} w_{ij}y_iy_j - \sum_i b_iy_i\right)$$

  - Just 0.5 times the negative of D
    - The 0.5 is only needed for convention
- The evolution of a Hopfield network constantly decreases its energy

Story so far

- A Hopfield network is a loopy binary network with symmetric connections
- Every neuron in the network attempts to “align” itself with the sign of the weighted combination of outputs of other neurons
  - The local “field”
- Given an initial configuration, neurons in the net will begin to “flip” to align themselves in this manner
  - Causing the field at other neurons to change, potentially making them flip
- Each evolution of the network is guaranteed to decrease the “energy” of the network
  - The energy is lower bounded and the decrements are upper bounded, so the network is guaranteed to converge to a stable state in a finite number of steps

Poll 1 @1794, @1795, @1796

Hopfield networks are loopy networks whose output activations "evolve" over time

- True
- False

Hopfield networks will evolve continuously, forever

- True
- False

Hopfield networks can also be viewed as infinitely deep shared parameter MLPs

- True
- False

Poll 1

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- **True**
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Hopfield networks will evolve continuously, forever

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- **False**

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The Energy of a Hopfield Net

- Define the *Energy* of the network as

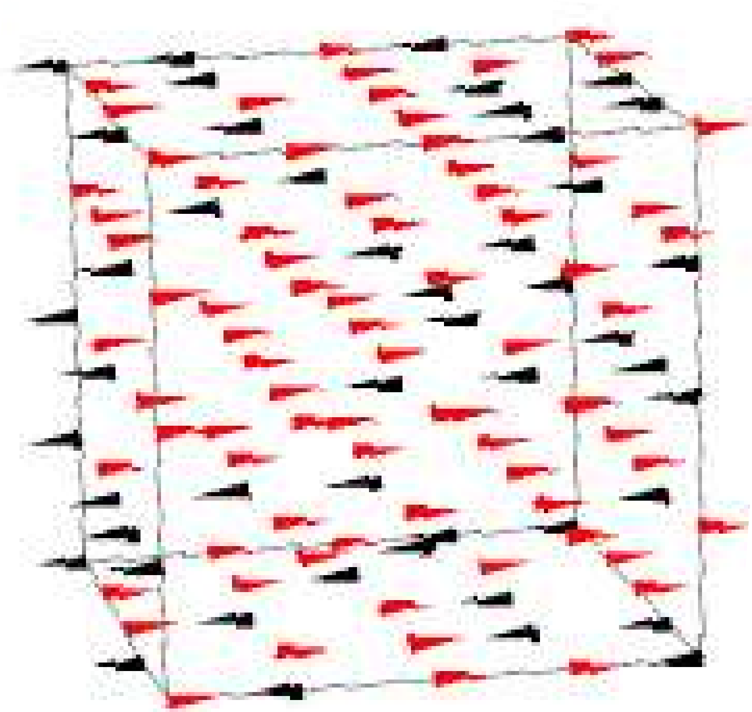
$$E = -\frac{1}{2}\left(\sum_{i,j\neq i} w_{ij} y_i y_j - \sum_i b_i y_i\right)$$

  - Just 0.5 times the negative of D

- The evolution of a Hopfield network constantly decreases its energy

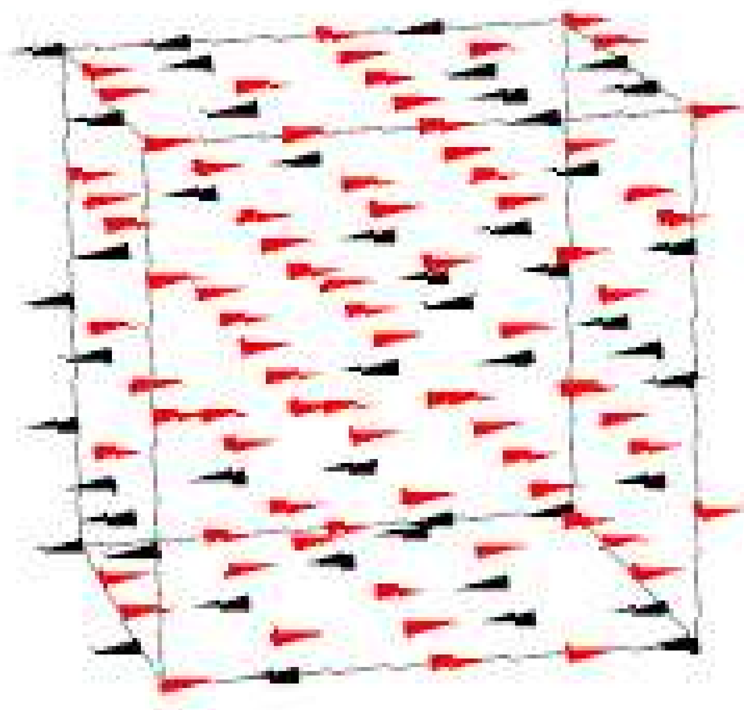
- Where did this “energy” concept suddenly sprout from?

Analogy: Spin Glass



- Magnetic diploes in a disordered magnetic material
- Each dipole tries to *align* itself to the local field
  - In doing so it may flip
- This will change fields at *other* dipoles
  - Which may flip
- Which changes the field at the current dipole…

Analogy: Spin Glasses



Total field at current dipole:

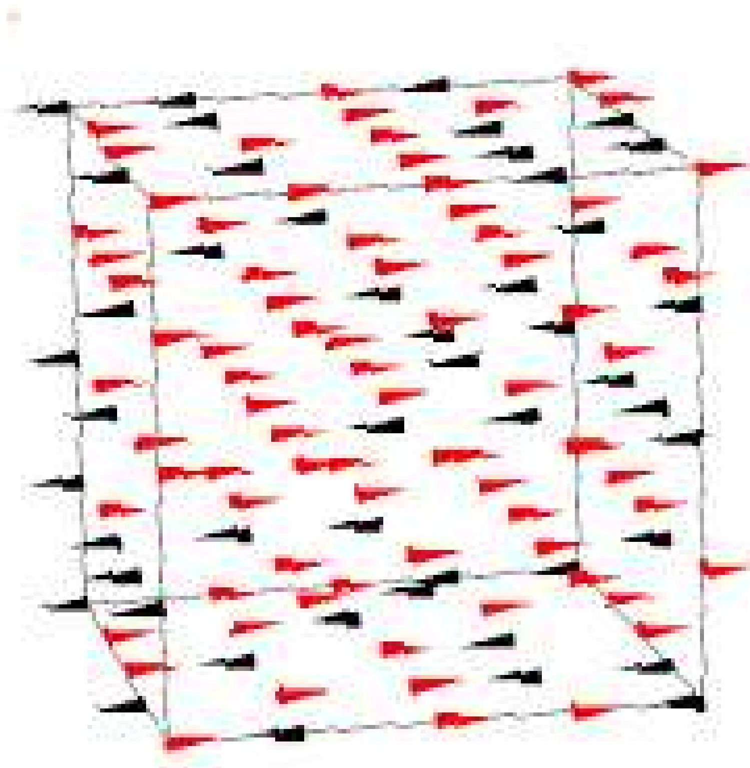
$$f(p_i) = \sum_{j \neq i} J_{ji} x_j + b_i$$

intrinsic

external

- p_i is vector position of i-th dipole
- The field at any dipole is the sum of the field contributions of all other dipoles
- The contribution of a dipole to the field at any point depends on interaction J
  - Derived from the "Ising" model for magnetic materials (Ising and Lenz, 1924)

Analogy: Spin Glasses



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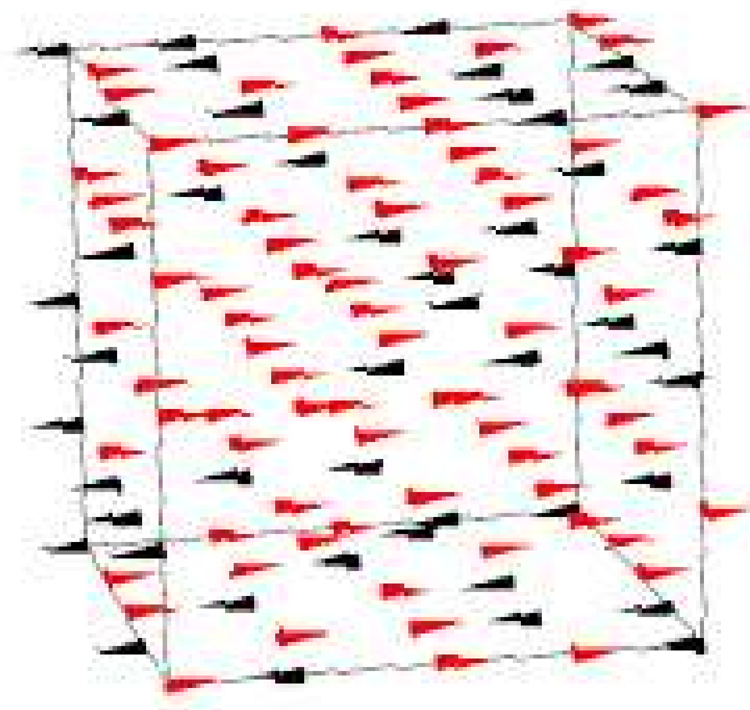
Response of current dipole

$$x_i = \begin{cases} x_i \ if \ sign\left(x_i \ f(p_i)\right) = 1 \\ -x_i \ otherwise \end{cases}$$

- A Dipole flips if it is misaligned with the field in its location

Analogy: Spin Glasses



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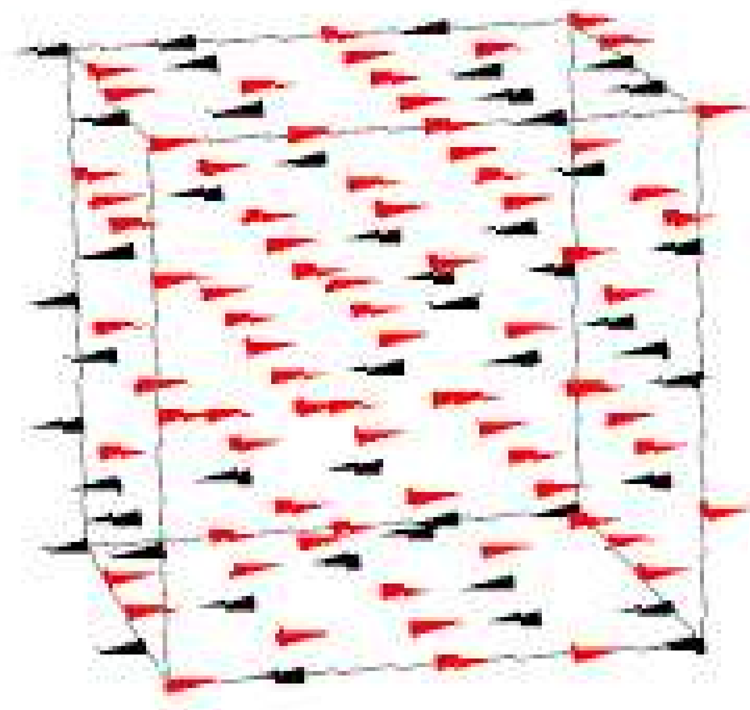
- Dipoles will keep flipping
  - A flipped dipole changes the field at other dipoles
    - Some of which will flip
  - Which will change the field at the current dipole
    - Which may flip
  - Etc..

Analogy: Spin Glasses



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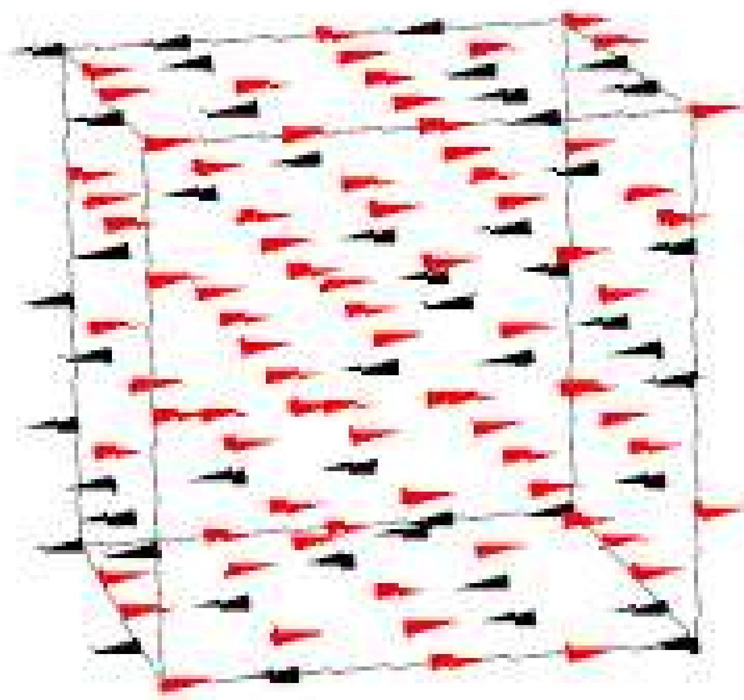
- When will it stop???

Analogy: Spin Glasses



Total field at current dipole:

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Response of current dipole

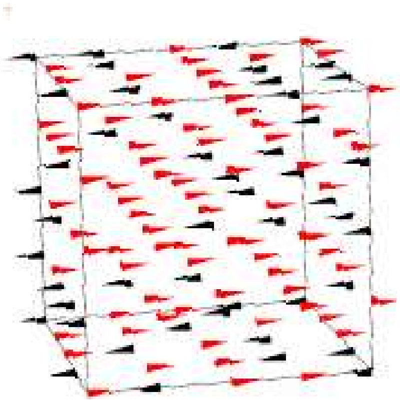
$$x_i = \begin{cases} x_i \ if \ sign\big(x_i \ f(p_i)\big) = 1 \\ -x_i \ otherwise \end{cases}$$

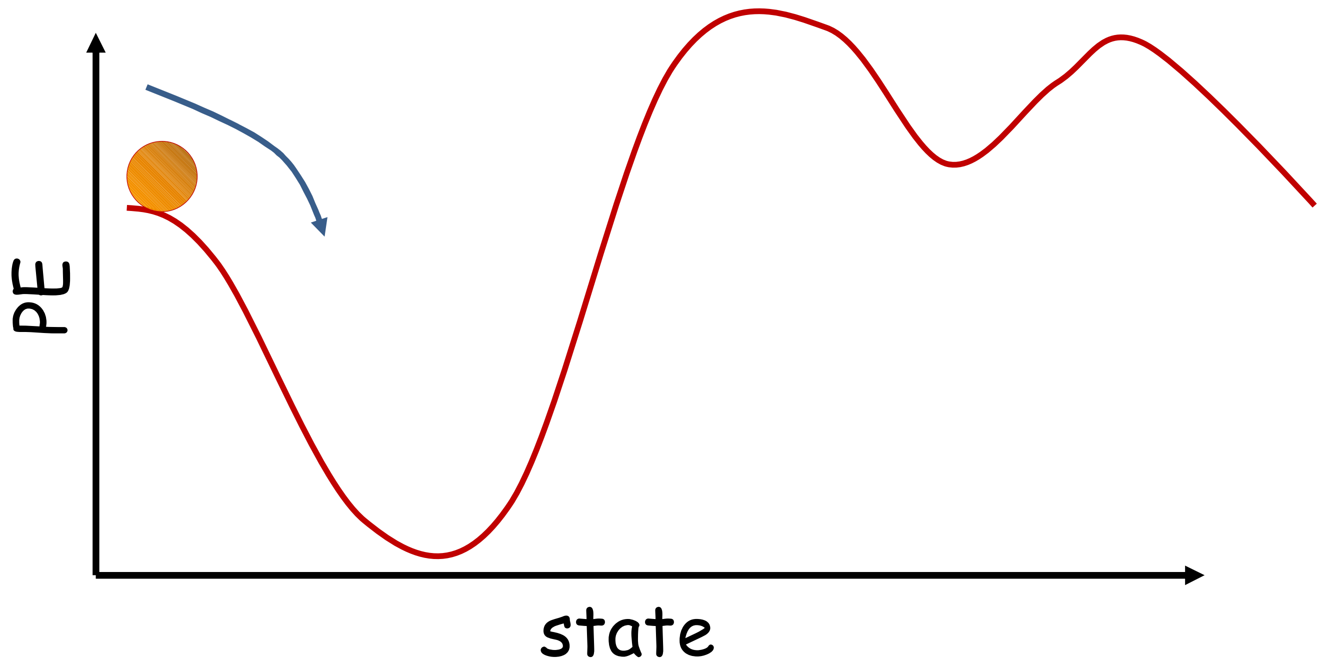
- The "Hamiltonian" (total energy) of the system

$$E = -\frac{1}{2} \sum_i x_i f(p_i) = -\sum_i \sum_{j>i} J_{ji} x_i x_j - \sum_i b_i x_i$$

- The system *evolves* to minimize the energy
  - Dipoles stop flipping if any flips result in increase of energy

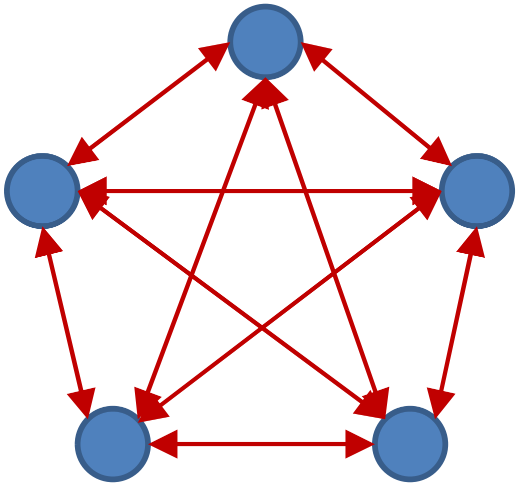
Spin Glasses





- The system stops at one of its *stable* configurations
  - Where energy is a local minimum
- Any small jitter from this stable configuration *returns it* to the stable configuration
  - I.e. the system *remembers* its stable state and returns to it

Hopfield Network



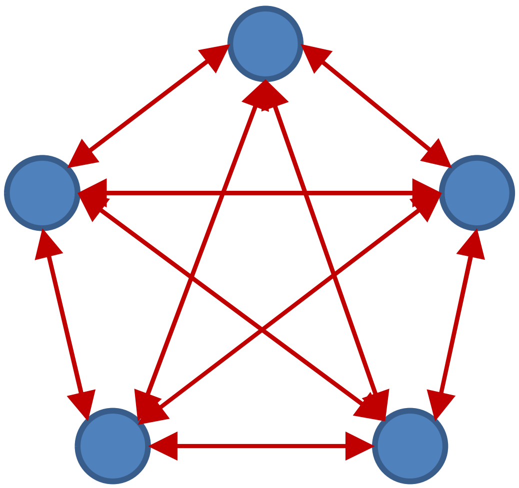
$$y_i = \Theta\left(\sum_{j \neq i} w_{ji} y_j + b_i\right)$$

$$\Theta(z) = \begin{cases} +1 \; if \; z > 0 \\ -1 \; if \; z \leq 0 \end{cases}$$

$$E = -\frac{1}{2}\left(\sum_{i,j \neq i} w_{ij} y_i y_j + \sum_i b_i y_i\right)$$

- This is analogous to the potential energy of a spin glass
  - The system will evolve until the energy hits a local minimum

Hopfield Network



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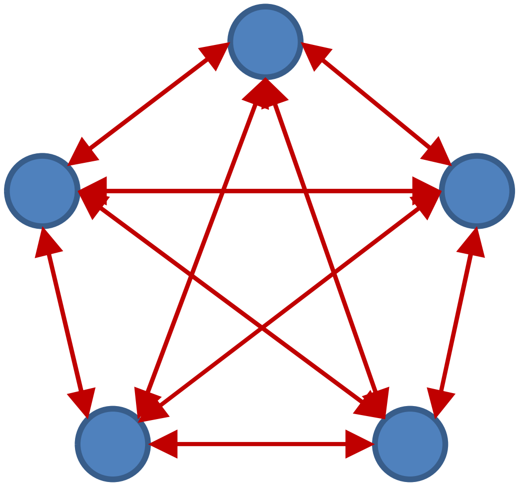
The bias is equivalent to having a single extra unit pegged at 1

We will not always explicitly show the bias

Often, in fact, a bias is not used, although in our case we are just being lazy in not showing it explicitly

Hopfield Network



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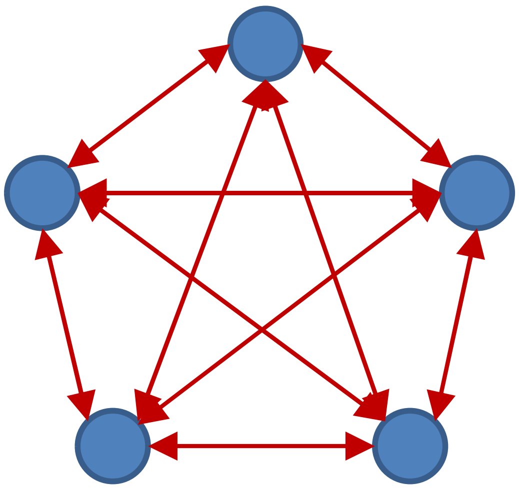
$$E = -\frac{1}{2} \sum_{i,j<i} w_{ij} y_i y_j$$

- This is analogous to the potential energy of a spin glass
  - The system will evolve until the energy hits a local minimum
    - Above equation is a factor of 0.5 off from earlier definition for conformity with thermodynamic system

Evolution



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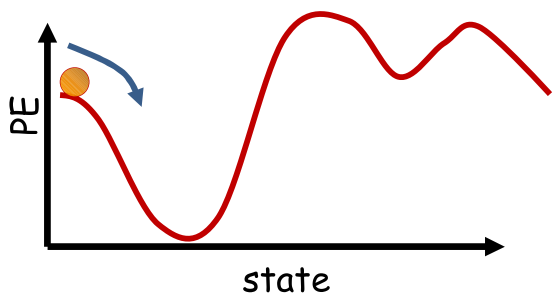
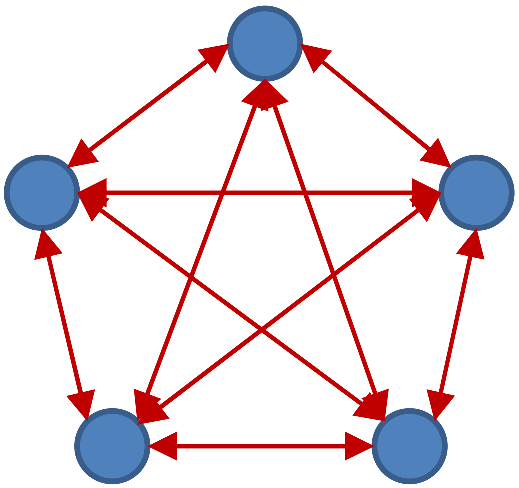
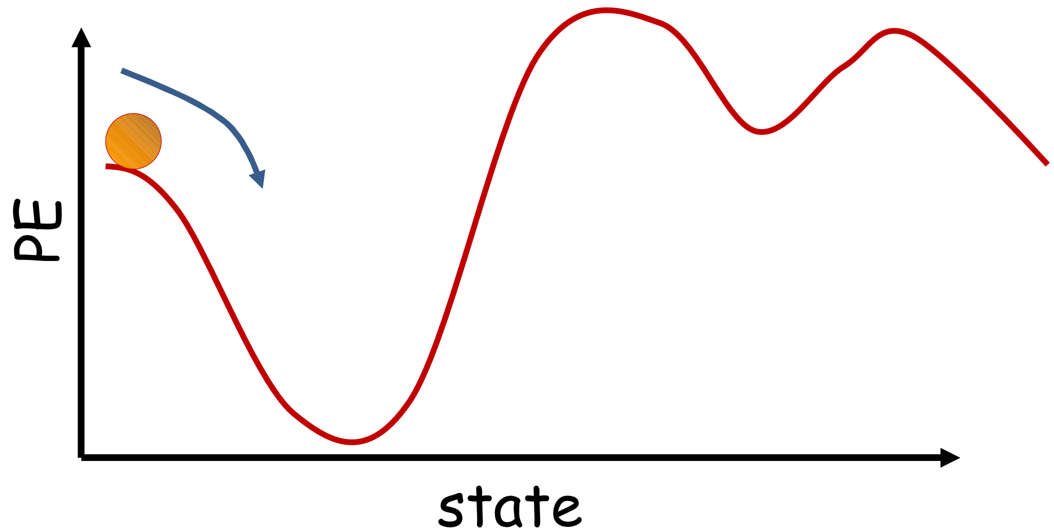


- The network will evolve until it arrives at a local minimum in the energy contour

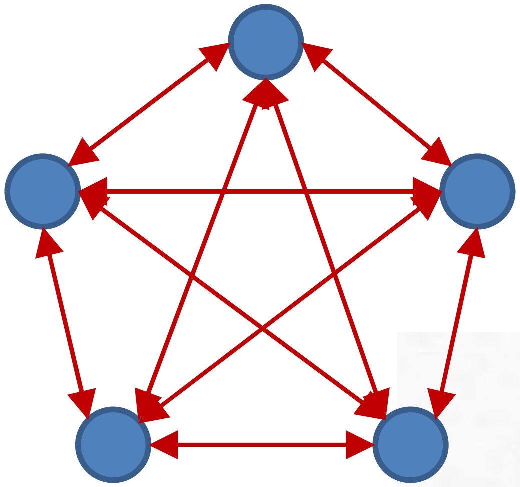
Content-addressable memory





- Each of the minima is a "stored" pattern
  - If the network is initialized close to a stored pattern, it will inevitably evolve to the pattern
- **This is a *content addressable memory***
  - Recall memory content from partial or corrupt values
- Also called ***associative memory***

Evolution



$$E = -\frac{1}{2}\sum_{i,j<i} w_{ij} y_i y_j$$

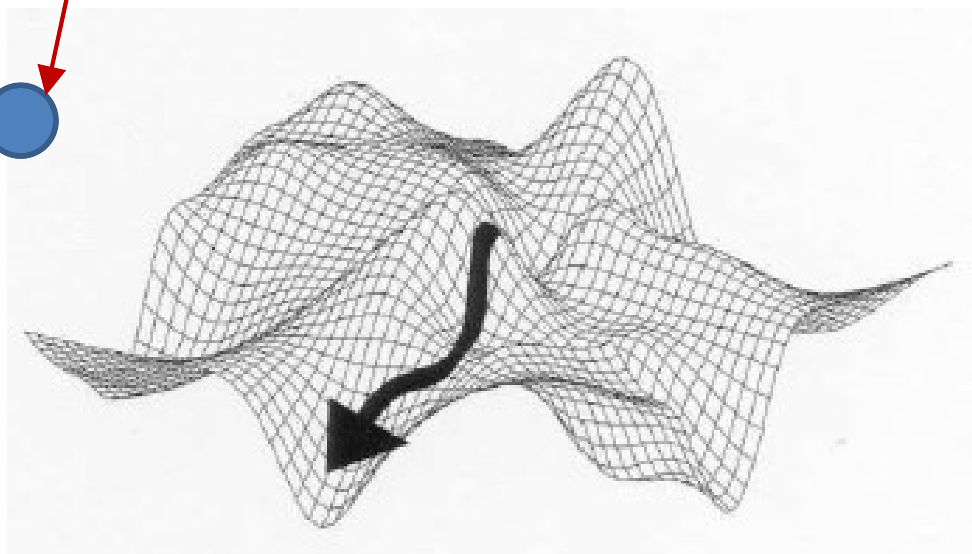
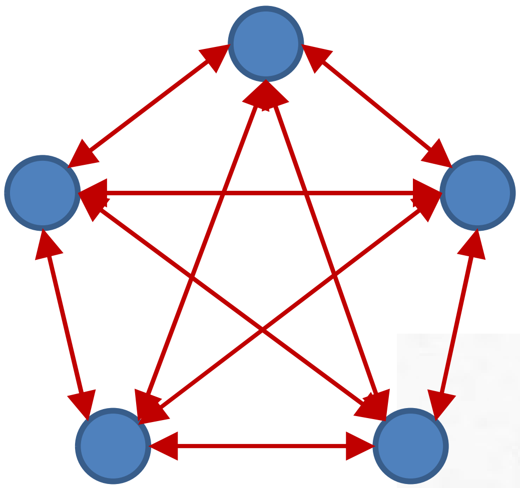


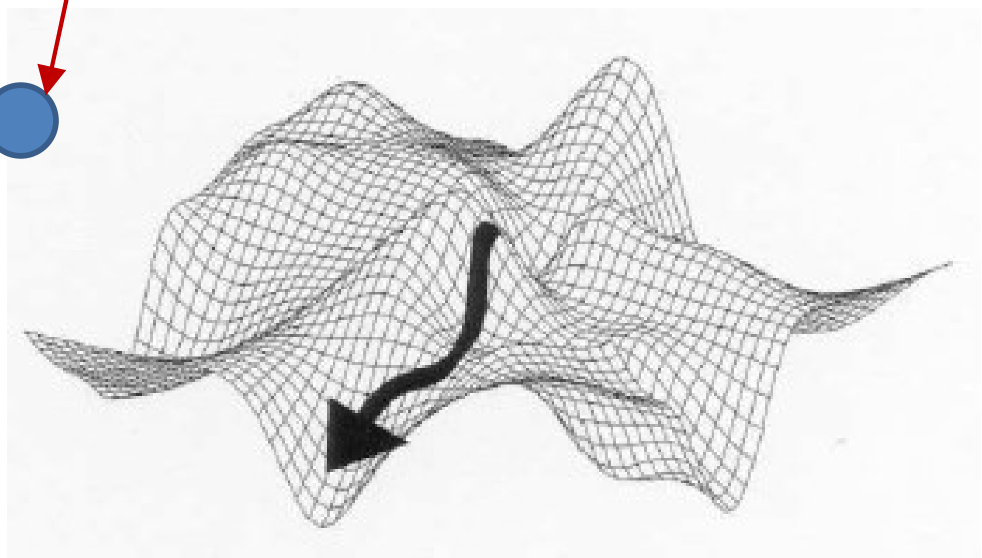
Image pilfered from unknown source

- The network will evolve until it arrives at a local minimum in the energy contour

Evolution

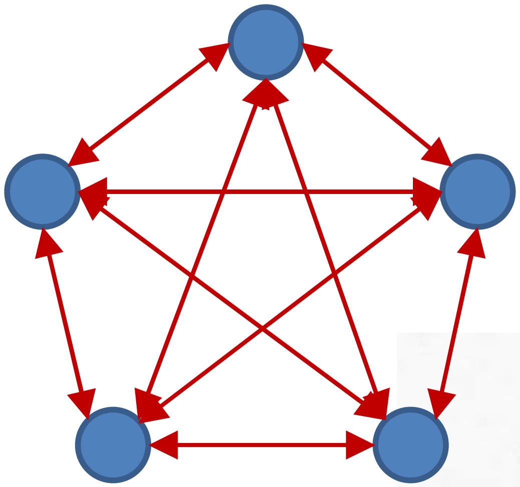


$$E = -\frac{1}{2}\sum_{i,j<i} w_{ij}y_iy_j$$



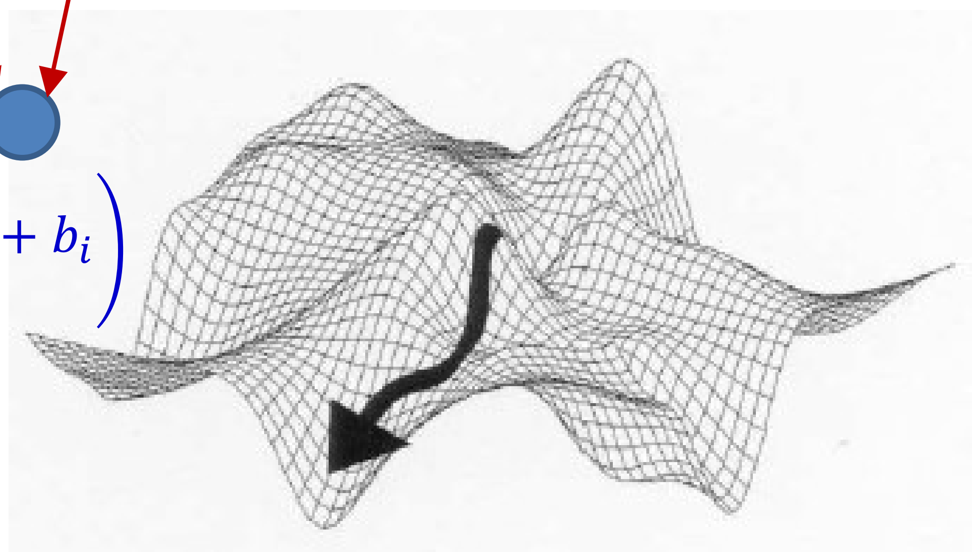
- The network will evolve until it arrives at a local minimum in the energy contour
- We proved that *every* change in the network will result in *decrease* in energy
  - So path to energy minimum is monotonic

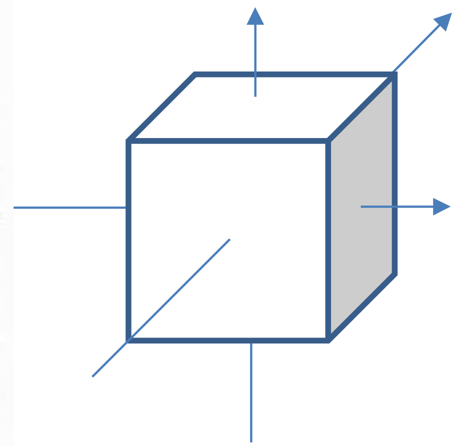
Evolution



$$E = -\frac{1}{2}\sum_{i,j<i} w_{ij}y_iy_j$$

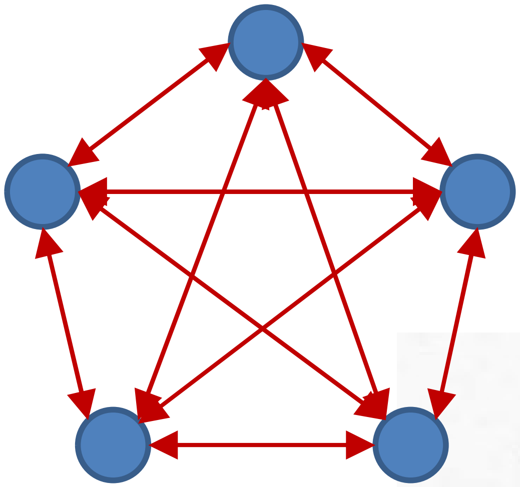
$$y_i = \Theta\left(\sum_{j\neq i} w_{ji}y_j + b_i\right)$$





- For threshold activations the energy contour is only defined on a lattice
  - Corners of a unit cube on $[-1,1]^N$

Evolution



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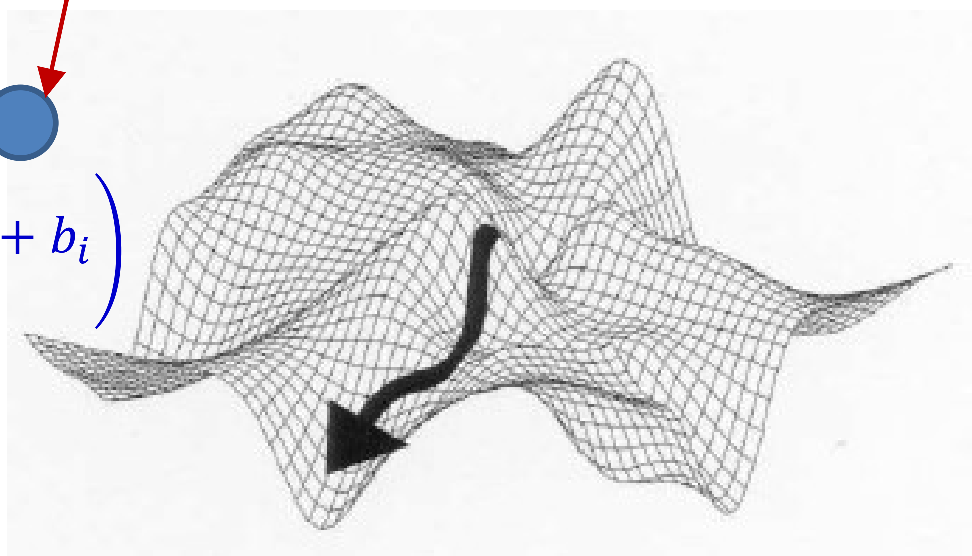
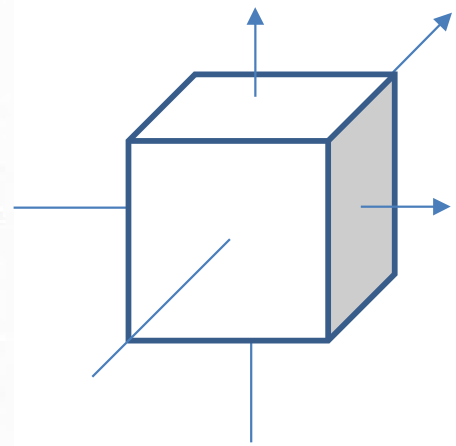
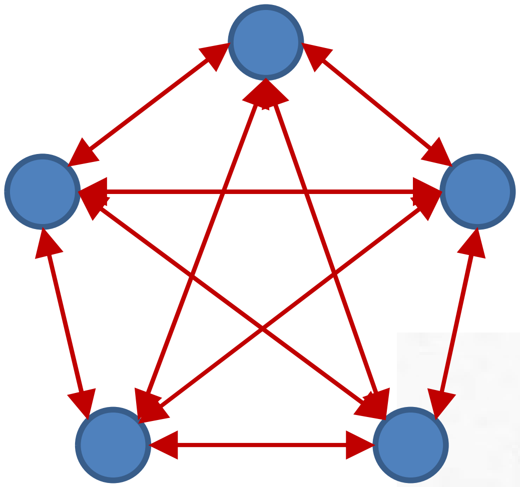




- For threshold activations the energy contour is only defined on a lattice
  - Corners of a unit cube on $[-1,1]^N$
- For tanh activations it will be a continuous function

Evolution



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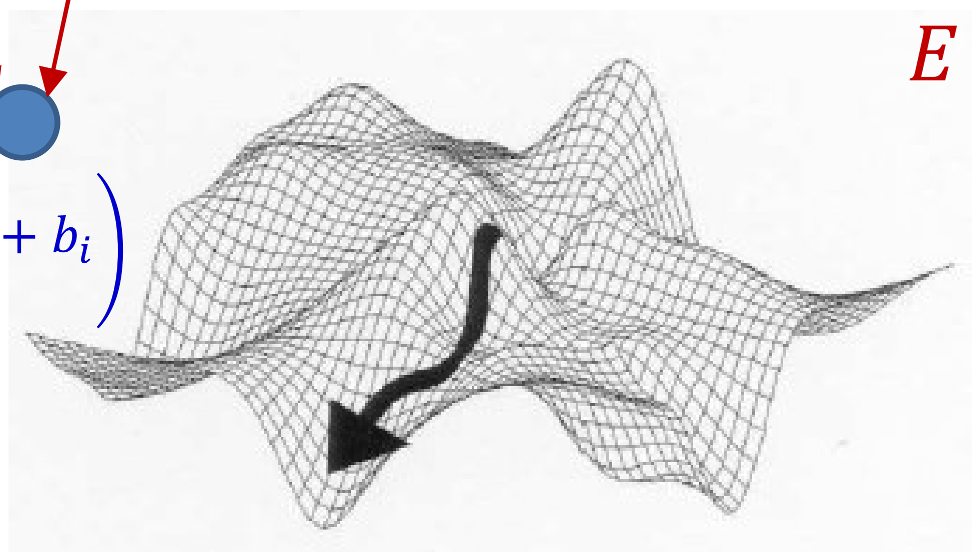
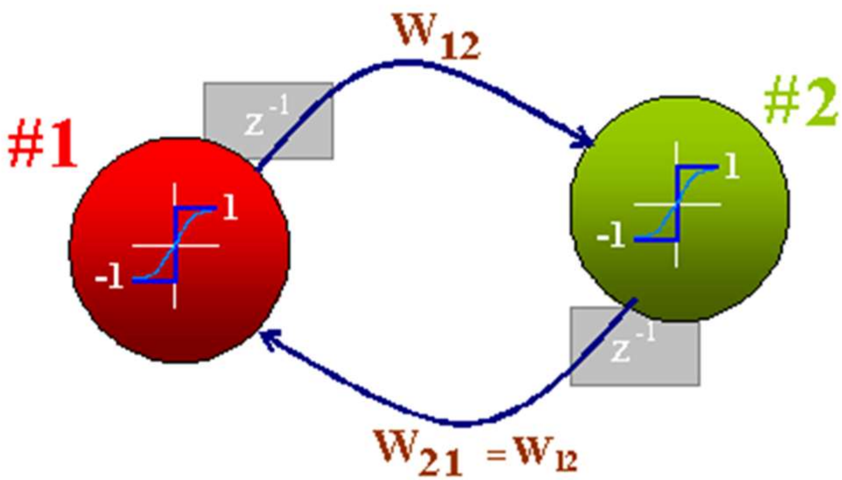
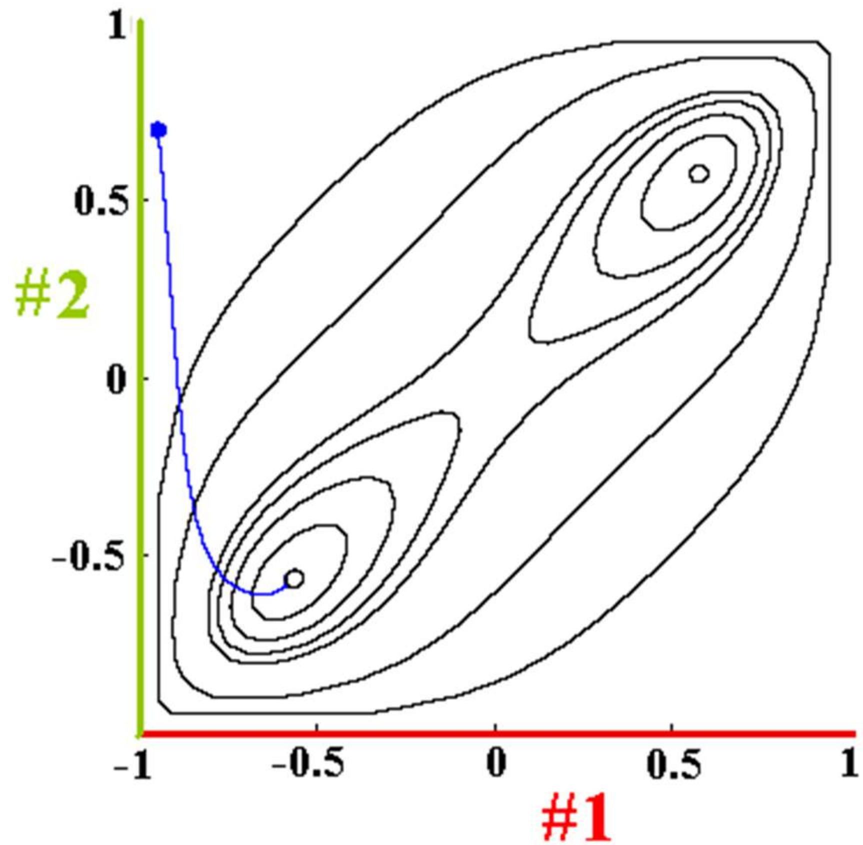


$$E = -\frac{1}{2}\mathbf{y}^T\mathbf{W}\mathbf{y}$$

- For threshold activations the energy contour is only defined on a lattice
  - Corners of a unit cube
- For tanh activations it will be a continuous function
  - With output in [-1 1]

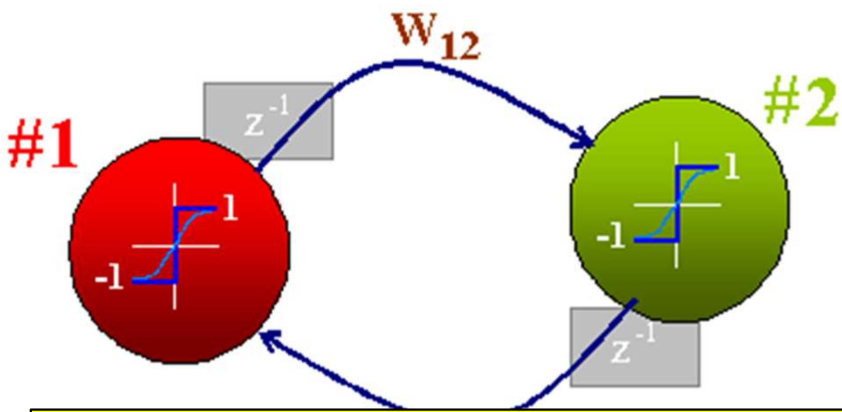
“Energy”contour for a 2-neuron net

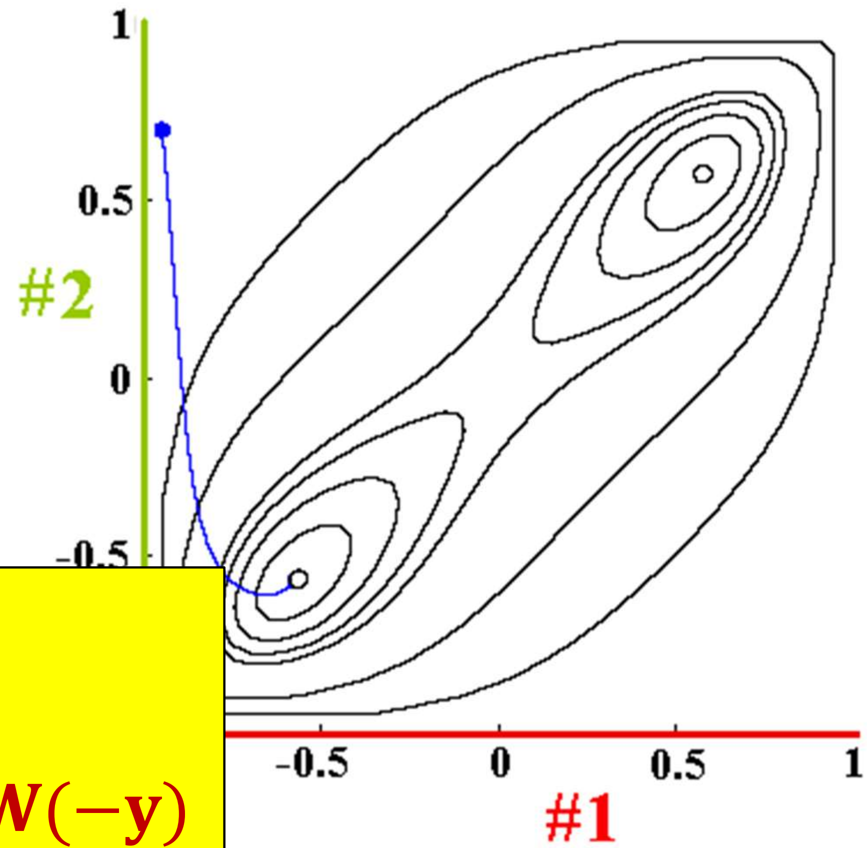




- Two stable states (tanh activation)
  - Symmetric, not at corners
  - Blue arc shows a typical trajectory for tanh activation

“Energy”contour for a 2-neuron net





Why symmetric?

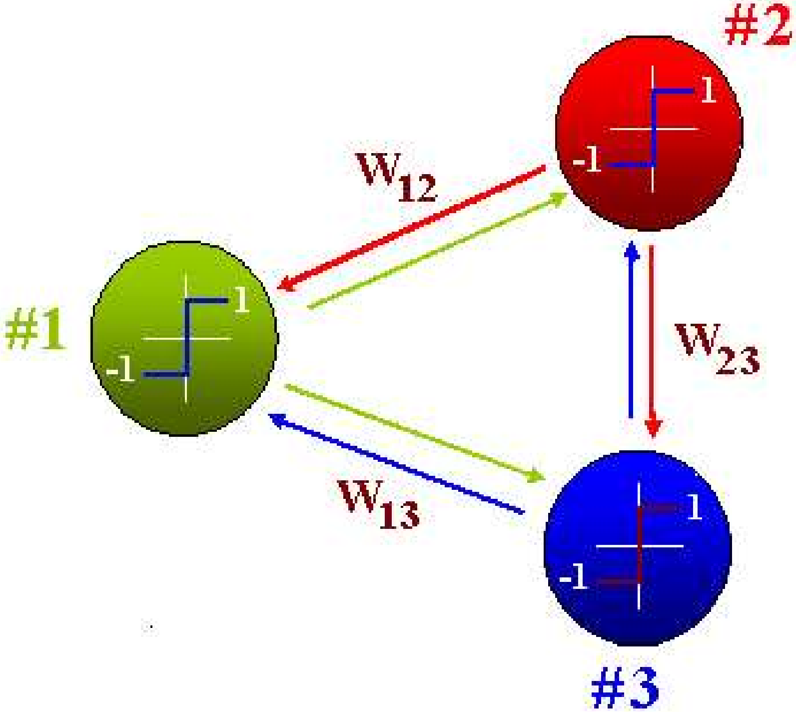
Because $-\frac{1}{2}\mathbf{y}^T\mathbf{W}\mathbf{y} = -\frac{1}{2}(-\mathbf{y})^T\mathbf{W}(-\mathbf{y})$

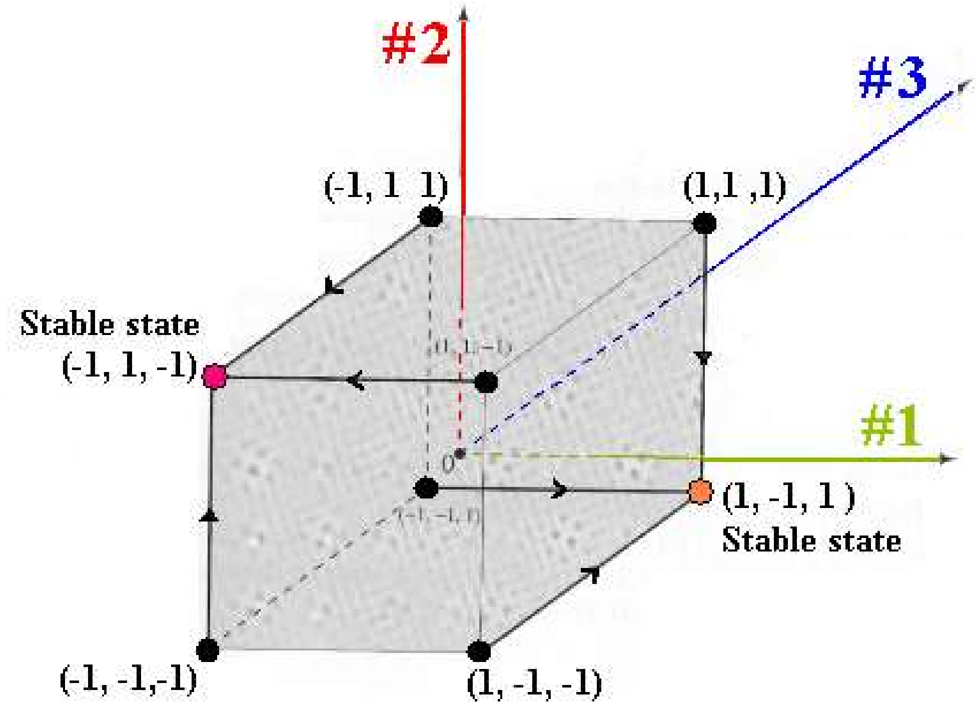
If $\hat{\mathbf{y}}$ is a local minimum, so is $-\hat{\mathbf{y}}$

•

– Blue arc shows a typical trajectory for sigmoid activation

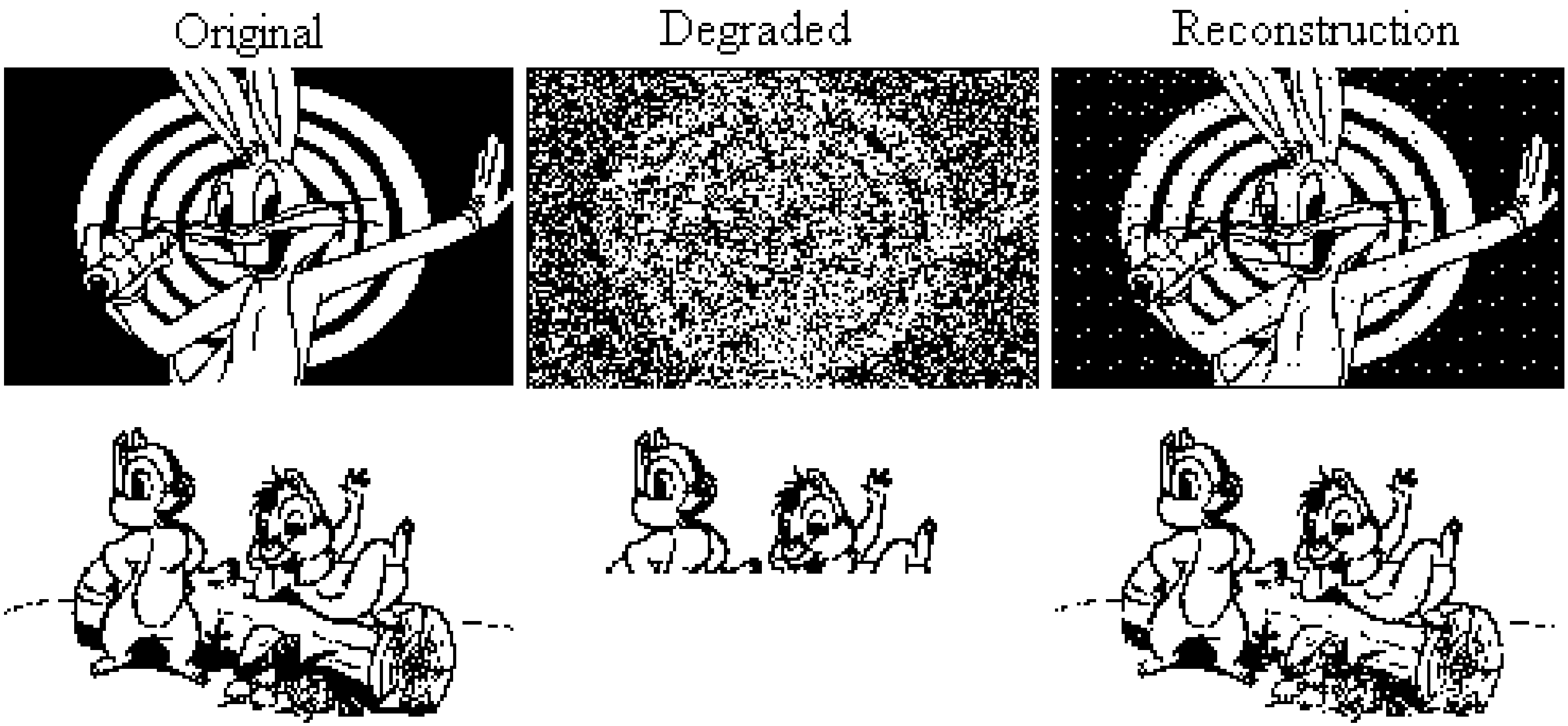
3-neuron net





- 8 possible states
- 2 stable states (hard thresholded network)

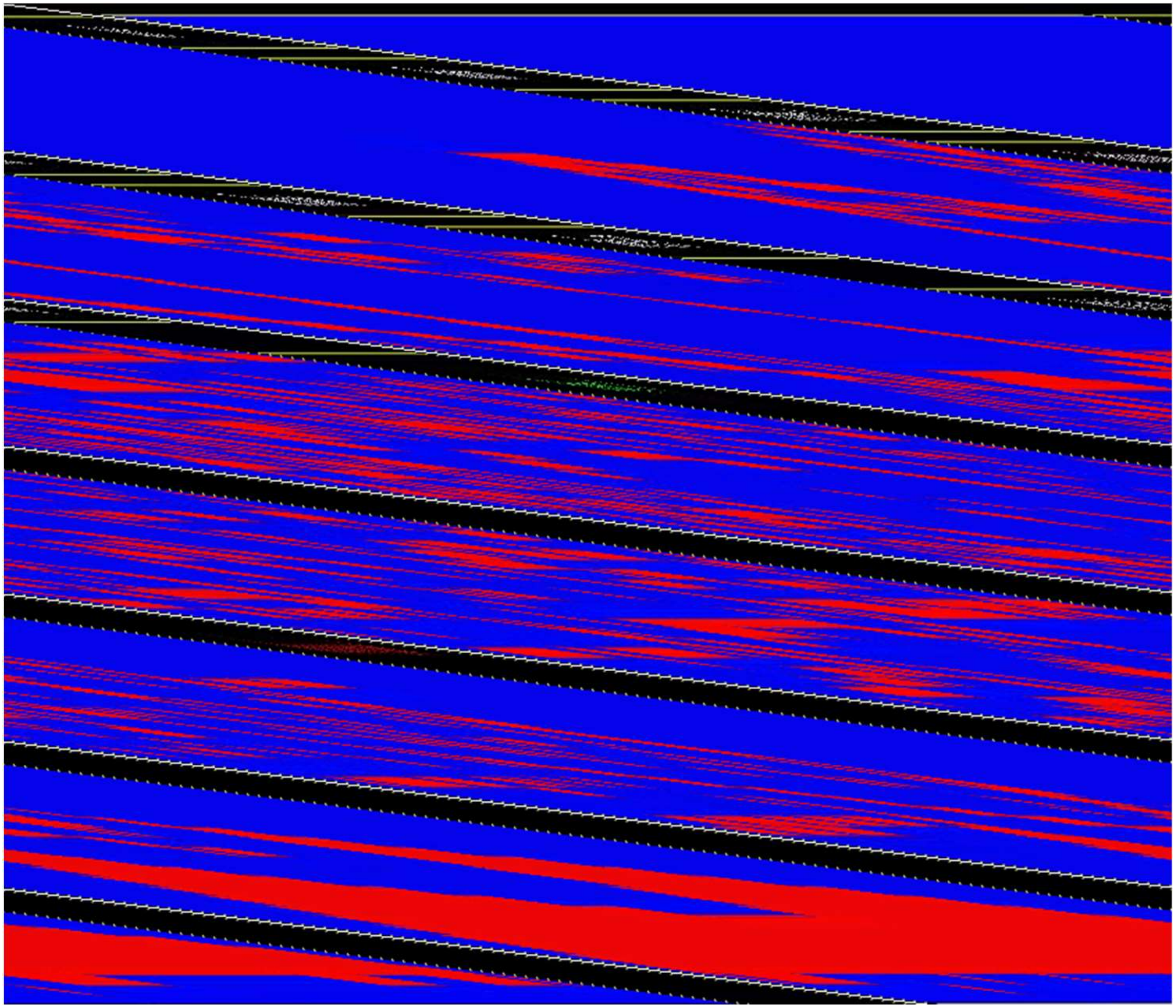
Examples: Content addressable memory



Hopfield network reconstructing degraded images from noisy (top) or partial (bottom) cues.

- http://staff.itee.uq.edu.au/janetw/cmc/chapters/Hopfield/

Hopfield net examples



Computational algorithm

1. Initialize network with initial pattern

$$y_i(0) = x_i, \qquad 0 \le i \le N-1$$

2. Iterate until convergence

$$y_i(t+1) = \Theta\left(\sum_{j \ne i} w_{ji} y_j\right), \qquad 0 \le i \le N-1$$

- Very simple
- Updates can be done sequentially, or all at once
- Convergence

$$E = -\sum_i \sum_{j>i} w_{ji} y_j y_i$$

does not change significantly any more

Computational algorithm

1. Initialize network with initial pattern

$$\mathbf{y} = \mathbf{x}, \qquad 0 \le i \le N-1$$

2. Iterate until convergence

$$\mathbf{y} = \Theta(\mathbf{W}\mathbf{y})$$

Writing $\mathbf{y} = [y_1, y_2, y_3, \cdots, y_N]^\top$
and arranging the weights as a matrix $\mathbf{W}$

- Very simple
- Updates can be done sequentially, or all at once
- Convergence

$$E = -0.5\mathbf{y}^\top\mathbf{W}\mathbf{y}$$

does not change significantly any more

Story so far

- A Hopfield network is a loopy binary network with symmetric connections
  - Neurons try to align themselves to the local field caused by other neurons
- Given an initial configuration, the patterns of neurons in the net will evolve until the "energy" of the network achieves a local minimum
  - The evolution will be monotonic in total energy
  - The dynamics of a Hopfield network mimic those of a spin glass
  - The network is symmetric: if a pattern Y is a local minimum, so is $-Y$
- The network acts as a *content-addressable* memory
  - If you initialize the network with a somewhat damaged version of a local-minimum pattern, it will evolve into that pattern
  - Effectively "recalling" the correct pattern, from a damaged/incomplete version

Poll 2, @1797

Mark all that are correct about Hopfield nets

- The network activations evolve until the energy of the net arrives at a local minimum
- Hopfield networks are a form of content addressable memory
- It is possible to analytically determine the stored memories by inspecting the weights matrix

Poll 2

Mark all that are correct about Hopfield nets

- **The network activations evolve until the energy of the net arrives at a local minimum**
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Issues

- How do we make the network store *a specific* pattern or set of patterns?
- How many patterns can we store?
- How to "retrieve" patterns better..

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- How do we make the network store *a specific* pattern or set of patterns?

- How many patterns can we store?

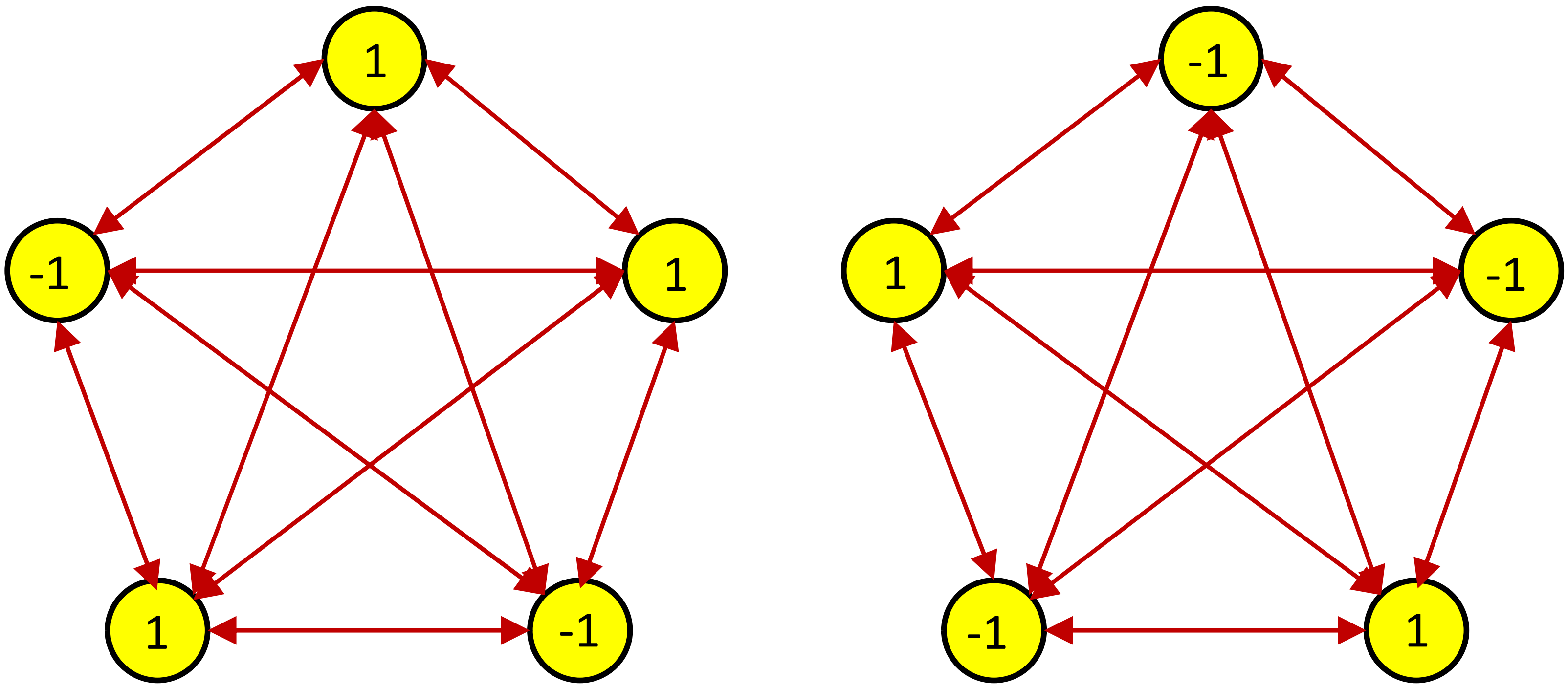
- How to "retrieve" patterns better..

How do we remember a *specific* pattern?

- How do we teach a network to "remember" this image

- For an image with N pixels we need a network with N neurons
- Every neuron connects to every other neuron
- Weights are symmetric (not mandatory)
- $\frac{N(N-1)}{2}$ weights in all

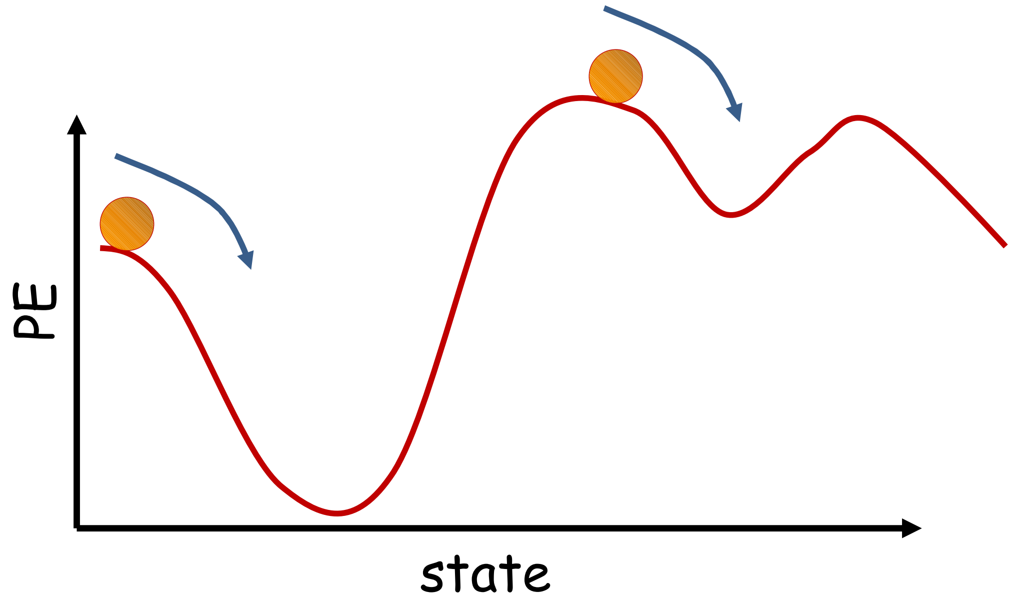
Storing patterns: Training a network

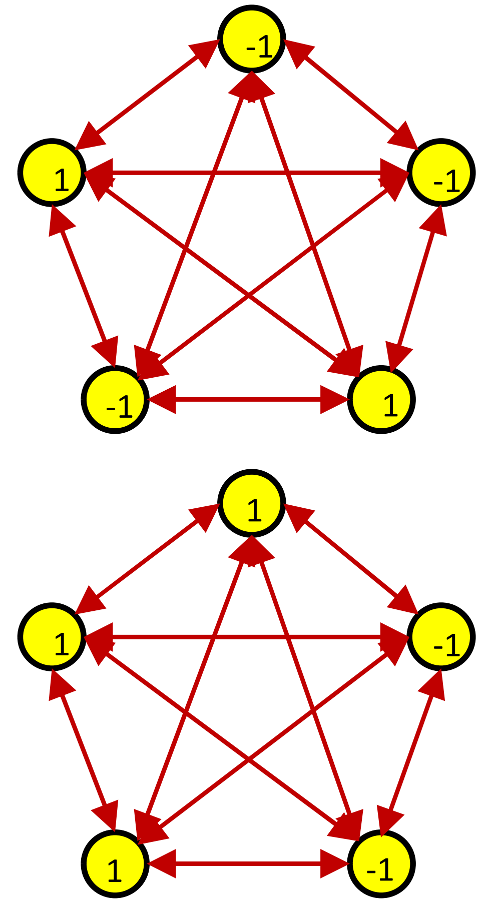


- A network that stores pattern P also naturally stores $-P$
  - Symmetry $E(P) = E(-P)$ since E is a function of $y_i y_j$

$$E = -\sum_i \sum_{j<i} w_{ji} y_j y_i$$

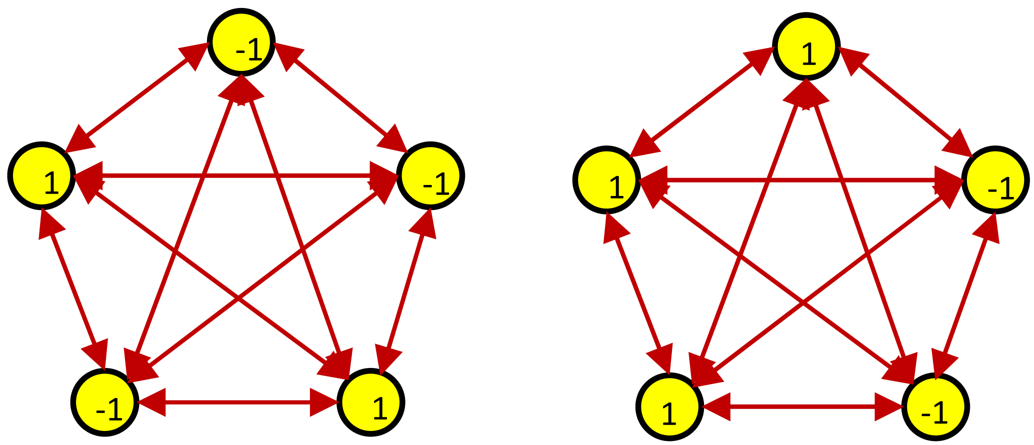
A network can store *multiple* patterns





- Every stable point is a stored pattern
- So we could design the net to store multiple patterns
  - Remember that every stored pattern P is actually *two* stored patterns, P and $-P$

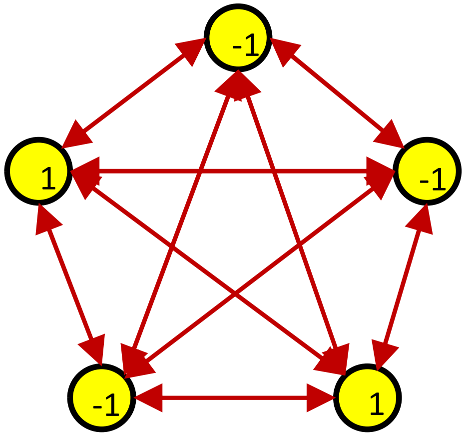
Storing a pattern



$$E = -\sum_{i}\sum_{j<i} w_{ji} y_j y_i$$

- Design $\{w_{ij}\}$ such that the energy is a local minimum at the desired $P = \{y_i\}$

Storing specific patterns

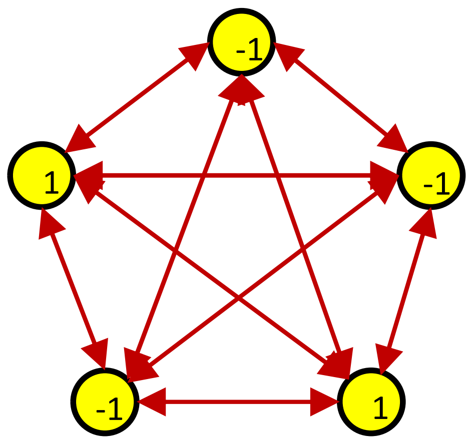


- Storing 1 pattern: We want

$$sign\left(\sum_{j \neq i} w_{ji} y_j\right) = y_i \quad \forall\ i$$

- This is a stationary pattern

Storing specific patterns



HEBBIAN LEARNING:

$$w_{ji} = y_j y_i$$

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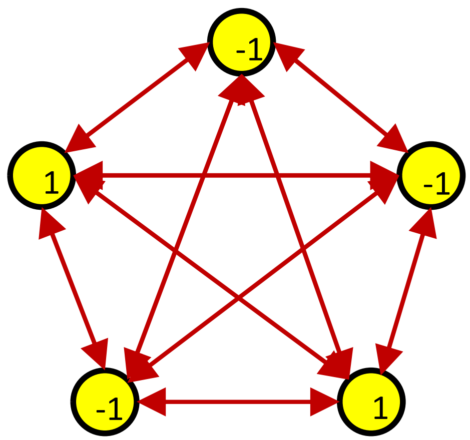

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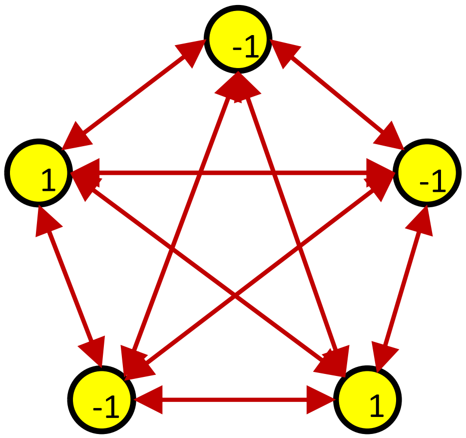
$$w_{ji} = y_j y_i$$

- $sign\left(\sum_{j \neq i} w_{ji} y_j\right) = sign\left(\sum_{j \neq i} y_j y_i y_j\right)$

$$= sign\left(\sum_{j \neq i} y_j^2 y_i\right) = sign(y_i) = y_i$$

Storing specific patterns



HEBBIAN LEARNING:

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The pattern is stationary

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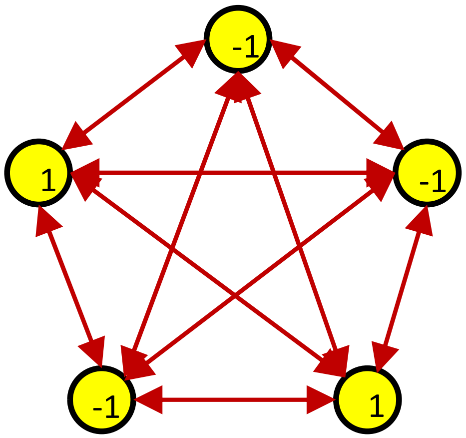


HEBBIAN LEARNING:

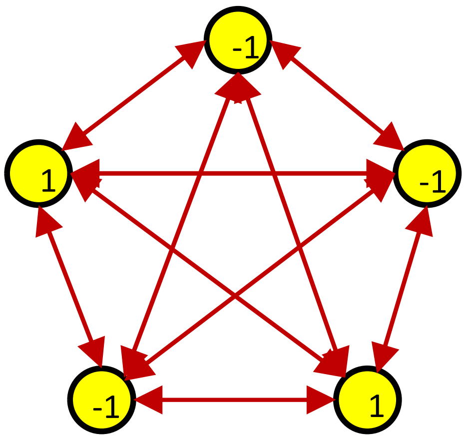
$$w_{ji} = y_j y_i$$

$$E = -\sum_i \sum_{j<i} w_{ji} y_j y_i = -\sum_i \sum_{j<i} y_i^2 y_j^2$$

$$= -\sum_i \sum_{j<i} 1 = -0.5N(N-1)$$

- This is the lowest possible energy value for the network for binary weights

Storing specific patterns



HEBBIAN LEARNING:

$$w_{ji} = y_j y_i$$

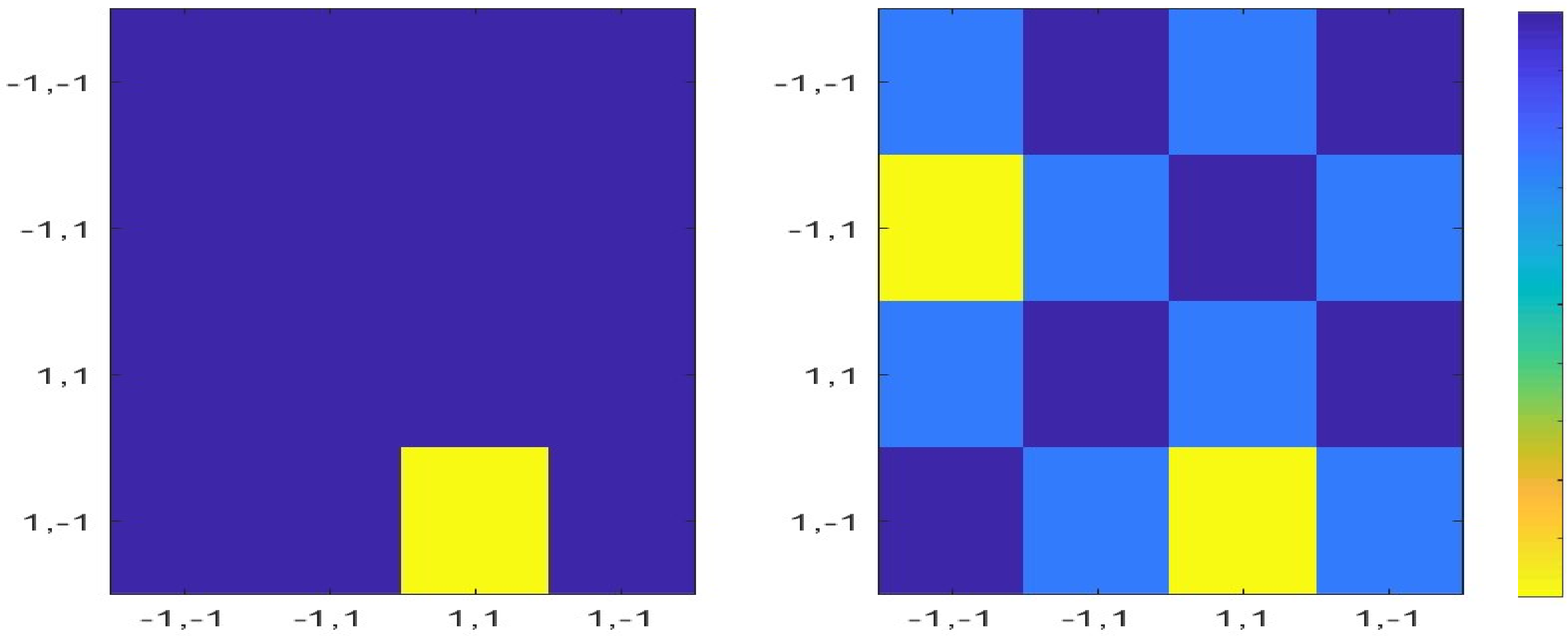
The pattern is *STABLE*

$$E = -\sum_i \sum_{j<i} w_{ji} y_j y_i = -\sum_i \sum_{j<i} y_i^2 y_j^2$$

$$= -\sum_i \sum_{j<i} 1 = -0.5N(N-1)$$

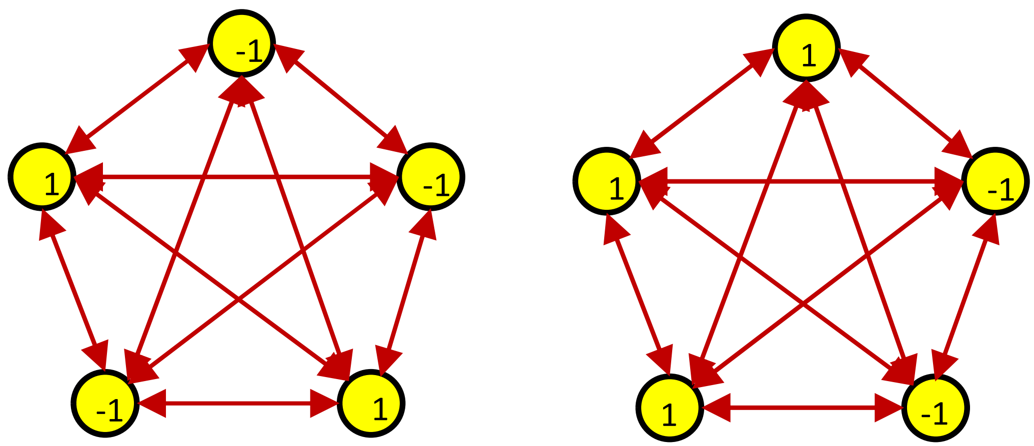
- This is the lowest possible energy value for the network for binary weights

Hebbian learning: Storing a 4-bit pattern



- Left: Pattern stored. Right: Energy map
- Stored pattern has lowest energy
- Gradation of energy ensures stored pattern (or its ghost) is recalled from everywhere

Storing multiple patterns

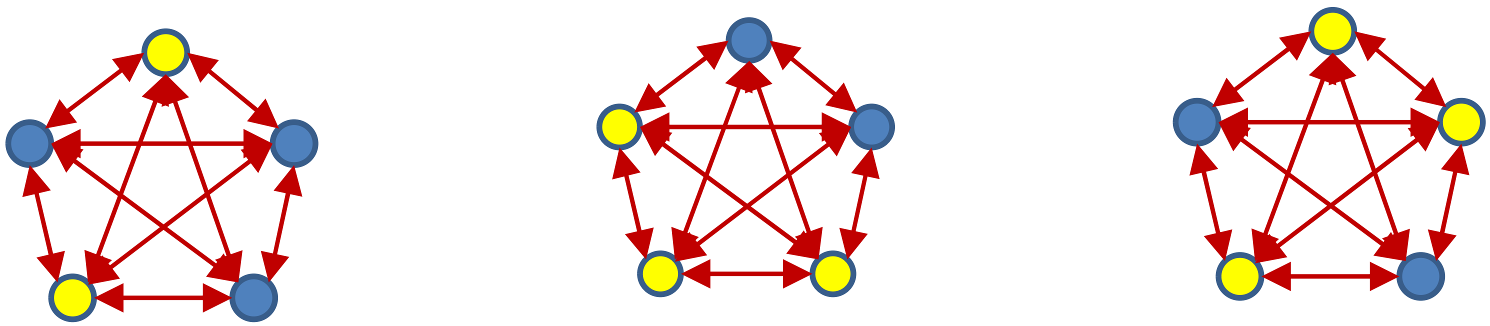


- To store *more* than one pattern

$$w_{ji} = \frac{1}{N} \sum_{\mathbf{y}_p \in \{\mathbf{y}_p\}} y_i^p y_j^p$$

- $\{\mathbf{y}_p\}$ is the set of patterns to store
- Super/subscript p represents the specific pattern
- N is the number of patterns

How many patterns can we store?



- **Hopfield**: For a network of N neurons can store up to ~$0.15N$ random patterns through Hebbian learning
  - Provided they are "far" enough
- Where did this number come from?

The limits of Hebbian Learning

- Consider the following: We must store K N-bit patterns of the form
$$\mathbf{y}_k = [y_1^k, y_2^k, \dots, y_N^k], k = 1 \dots K$$

- Hebbian learning (scaling by $\frac{1}{N}$ for normalization, this does not affect actual pattern storage):

$$w_{ij} = \frac{1}{N}\sum_k y_i^k y_j^k$$

- **For any pattern $\mathbf{y}_p$ to be stable:**

$$y_i^p \sum_j w_{ij} y_j^p > 0 \quad \forall i$$

$$y_i^p \frac{1}{N} \sum_j \sum_k y_i^k y_j^k \, y_j^p > 0 \quad \forall i$$

The limits of Hebbian Learning

- **For any pattern $\mathbf{y}_p$ to be stable:**

$$y_i^p \frac{1}{N} \sum_j \sum_k y_i^k y_j^k \, y_j^p > 0 \quad \forall i$$

$$y_i^p \frac{1}{N} \sum_j y_i^p y_j^p y_j^p + y_i^p \frac{1}{N} \sum_j \sum_{k \neq p} y_i^k y_j^k \, y_j^p > 0 \quad \forall i$$

- Note that the first term equals 1 (because $y_j^p y_j^p = y_i^p y_i^p = 1$)
  - i.e. for $\mathbf{y}_p$ to be stable the requirement is that the second *crosstalk term*:

$$y_i^p \frac{1}{N} \sum_j \sum_{k \neq p} y_i^k y_j^k \, y_j^p > -1 \quad \forall i$$

- The pattern will *fail* to be stored if the *crosstalk*

$$y_i^p \frac{1}{N} \sum_j \sum_{k \neq p} y_i^k y_j^k \, y_j^p < -1 \quad for\ any\ i$$

The limits of Hebbian Learning

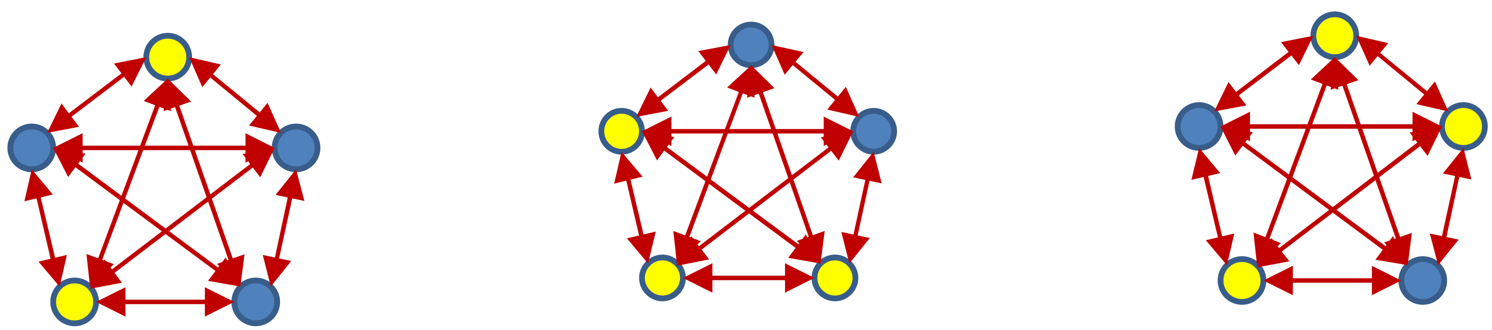
- For any random set of K patterns to be stored, the probability of the following must be low

$$\left(C_i^p = \frac{1}{N}\sum_j \sum_{k \neq p} y_i^p y_i^k y_j^k \, y_j^p\right) < -1$$

- For large N and K the probability distribution of C_i^p approaches a Gaussian with 0 mean, and variance K/N
  - Considering that individual bits $y_i^l \in \{-1, +1\}$ and have variance 1
- For a Gaussian, $C \sim N(0, K/N)$
  - $P(C < -1 \mid \mu = 0, \sigma^2 = K/N) < 0.004$ for $K/N < 0.14$

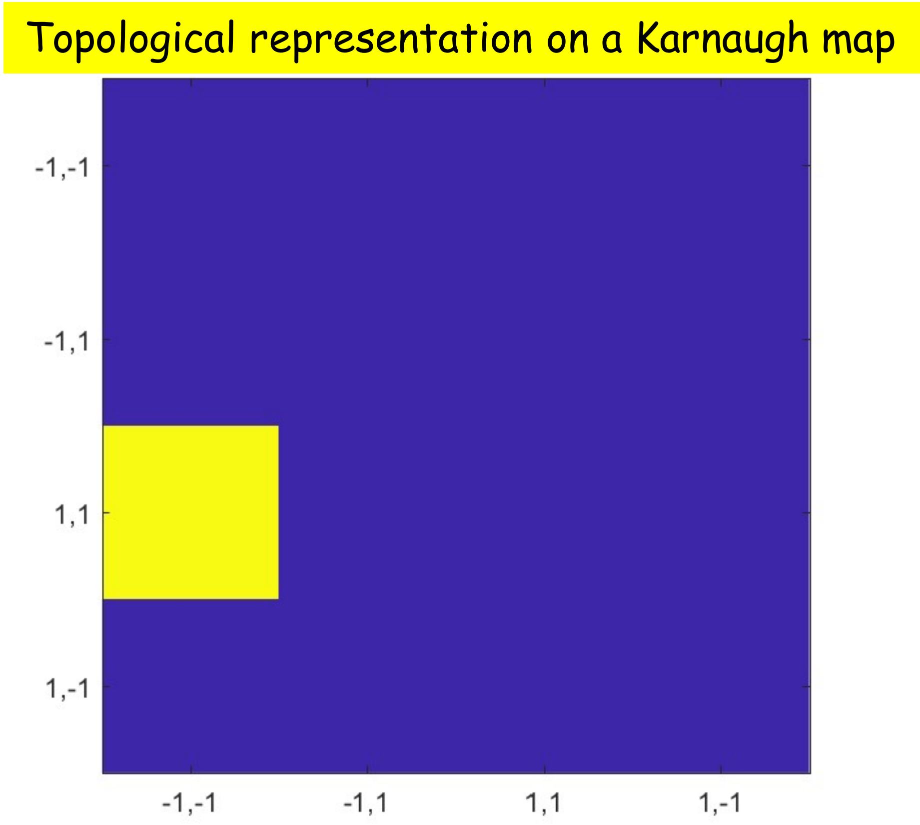
- I.e. To have less than 0.4% probability that stored patterns will *not* be stable, $K < 0.14N$

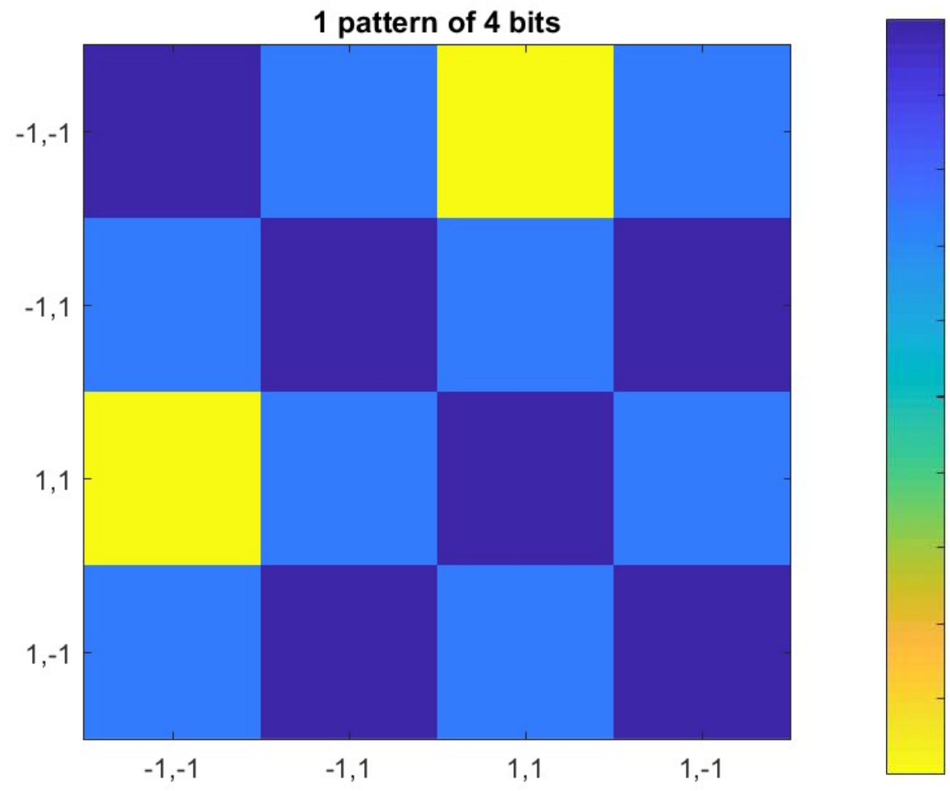
How many patterns can we store?



- A network of N neurons trained by Hebbian learning can store up to ~$0.14N$ random patterns with low probability of error
  - Computed assuming $prob(bit = 1) \;=\; 0.5$
    - On average no. of matched bits in any pair = no. of mismatched bits
      - Patterns are "orthogonal" – maximally distant – from one another
  - Expected behavior for *non-orthogonal* patterns?
- To get some insight into what is stored, lets see some examples

Hebbian learning: One 4-bit pattern





- Left: Pattern stored. Right: Energy map
- Note: Pattern is an energy well, but there are other local minima
  - Where?
  - Also note “shadow” pattern

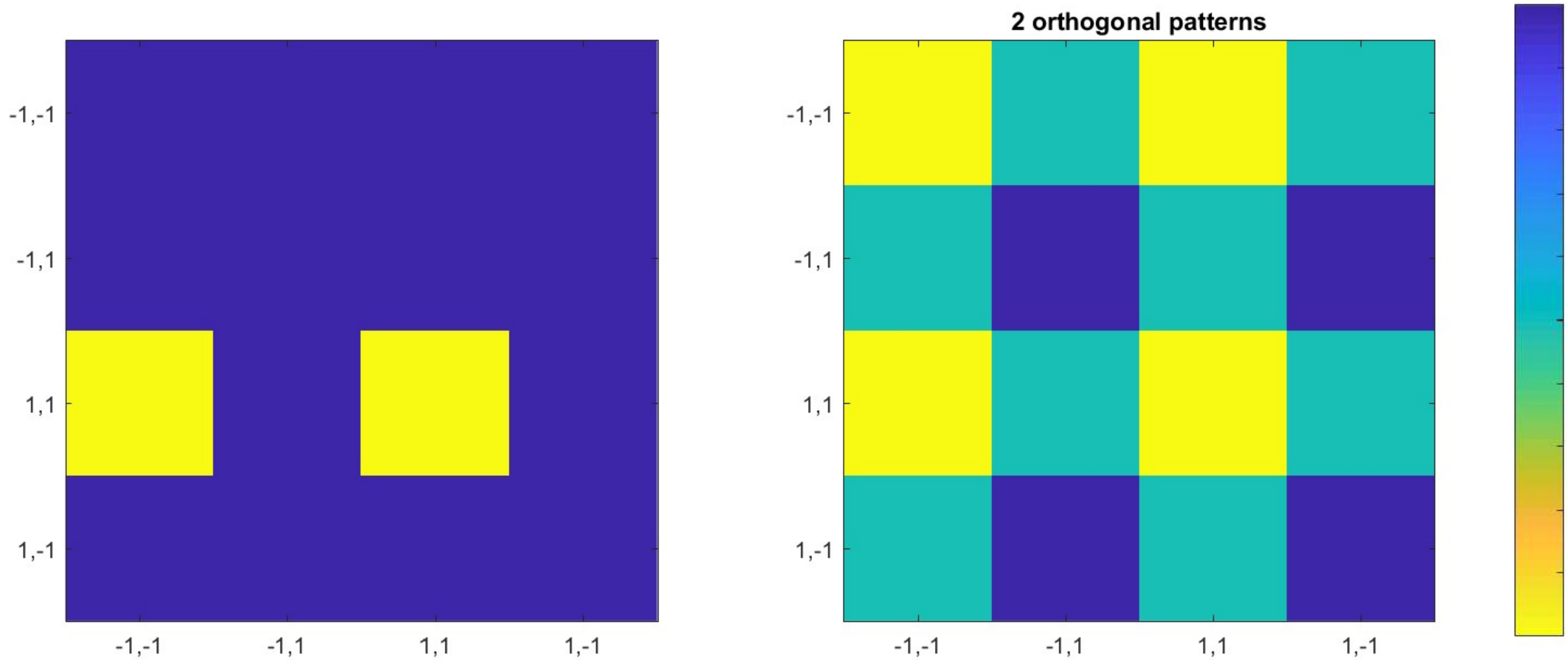
Storing multiple patterns: Orthogonality

- The maximum Hamming distance between two N-bit patterns is $N/2$
  - Because any pattern $Y = -Y$ for our purpose

- Two patterns y_1 and y_2 that differ in $N/2$ bits are *orthogonal*
  - Because $y_1^T y_2 = 0$

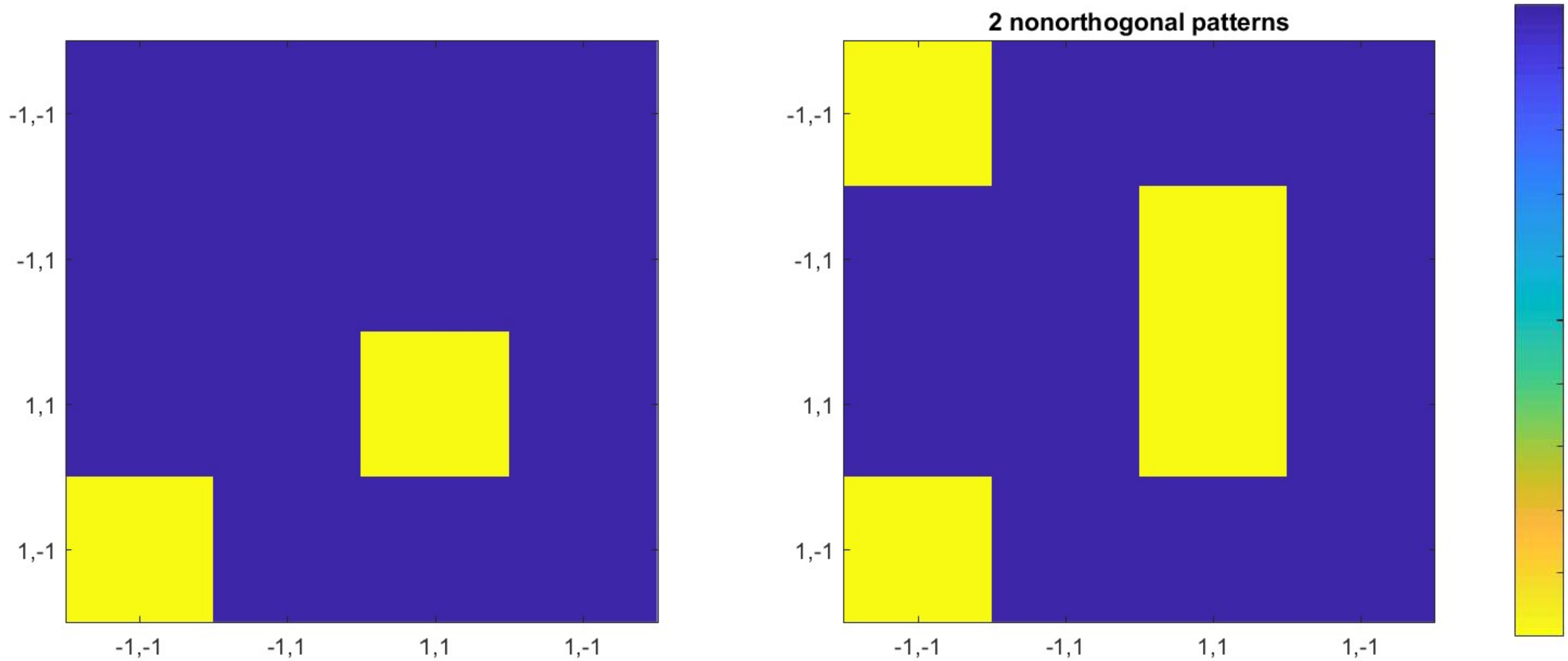
- For $N = 2^M L$, where L is an odd number, there are at most 2^M orthogonal binary patterns
  - Others may be *almost* orthogonal

Two orthogonal 4-bit patterns



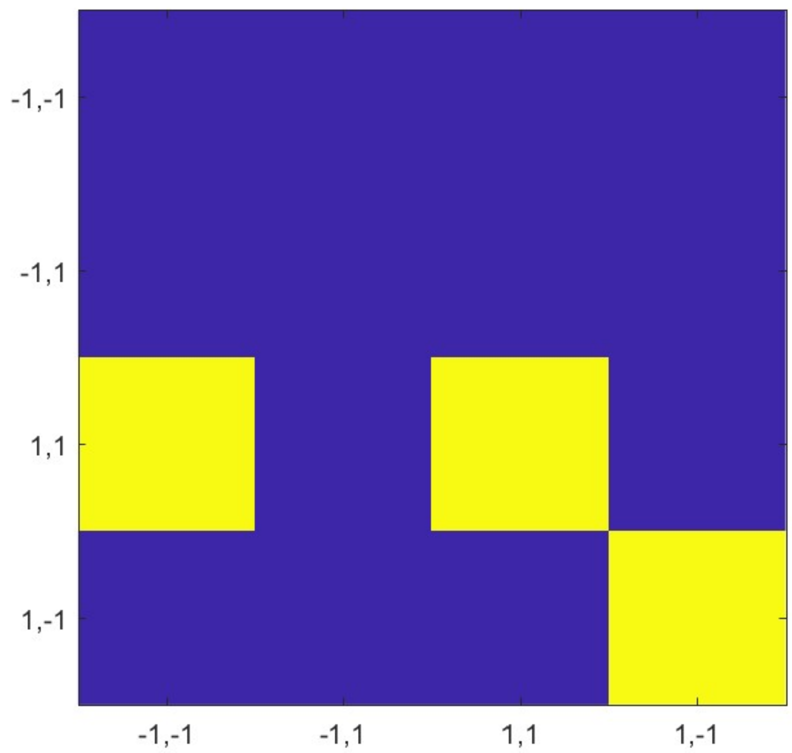
- Patterns are local minima (stationary and stable)
  - No other local minima exist
  - But patterns perfectly confusable for recall

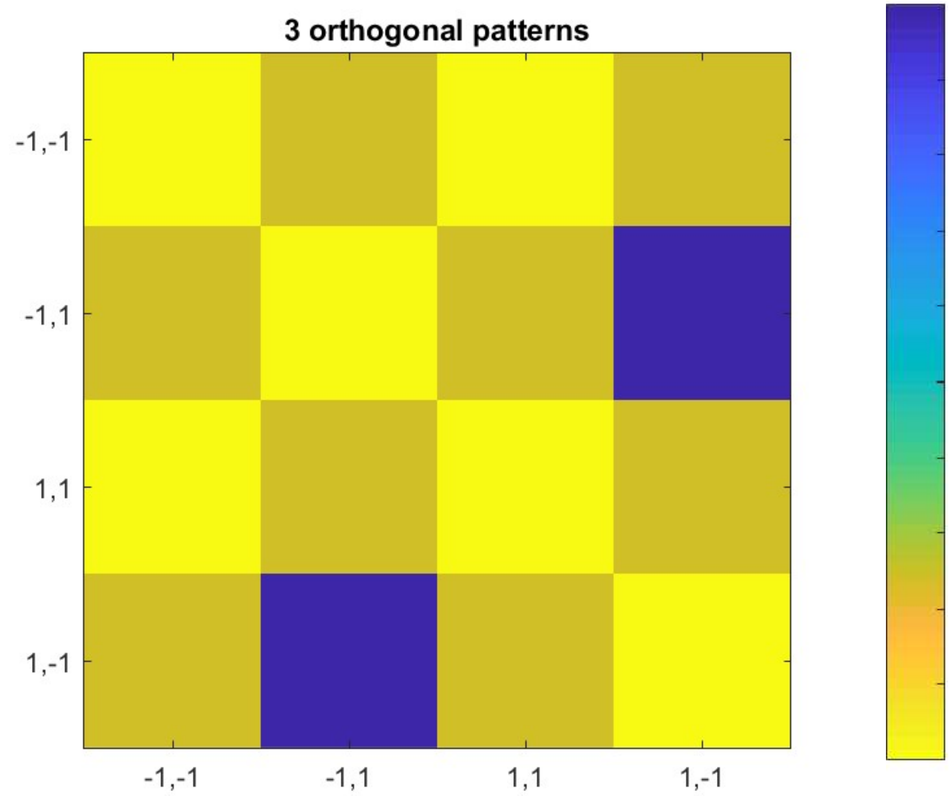
Two *non*-orthogonal 4-bit patterns



- Patterns are local minima (stationary and stable)
  - No other local minima exist
  - Actual *wells* for patterns
    - Patterns may be perfectly recalled!
  - Note K > 0.14 N

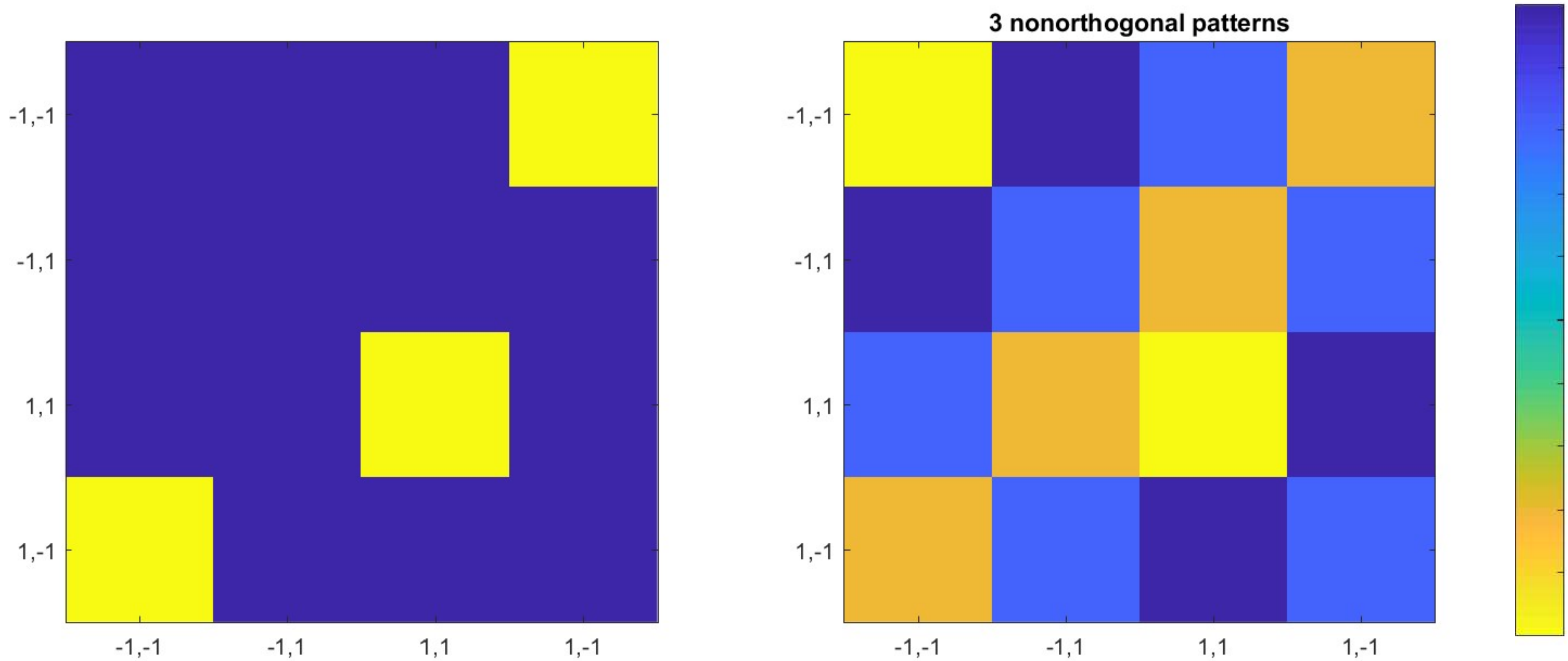
Three orthogonal 4-bit patterns





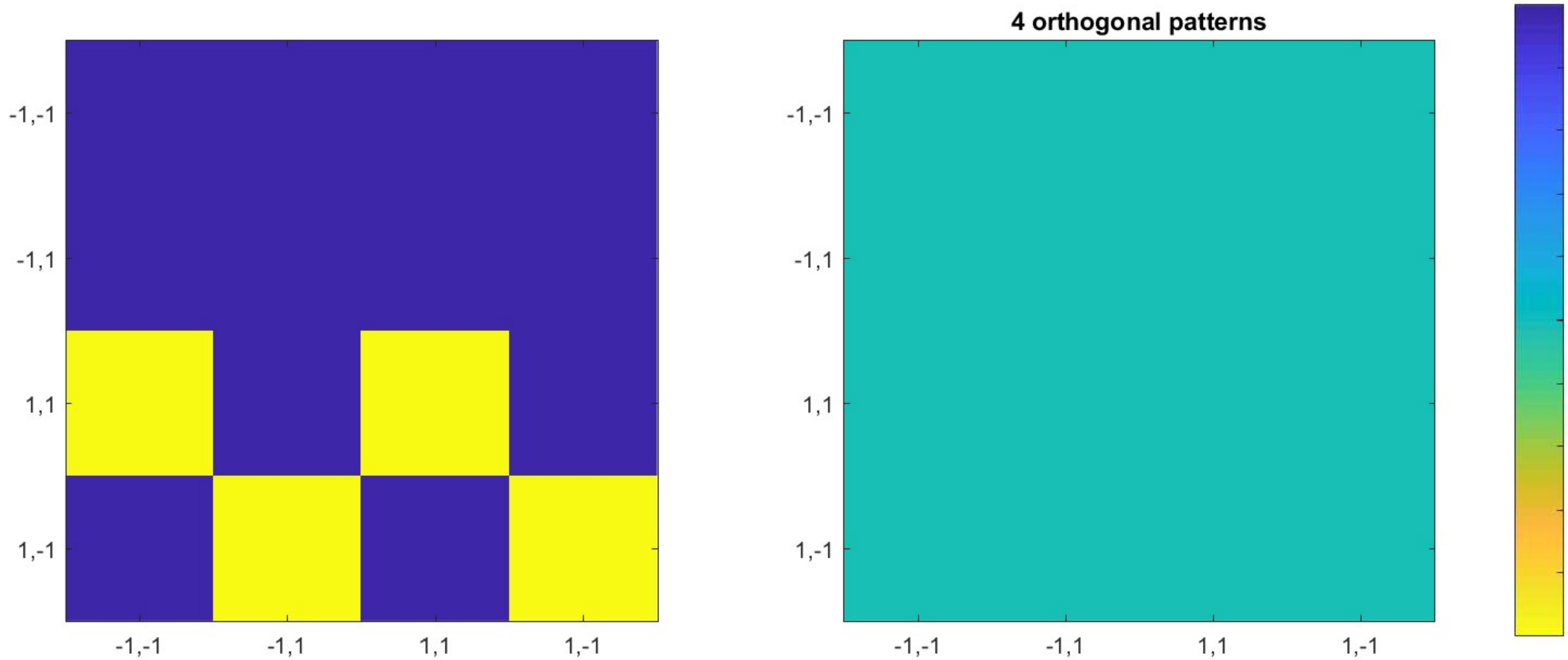
- All patterns are local minima (stationary)
  - – But recall from perturbed patterns is random

Three *non*-orthogonal 4-bit patterns



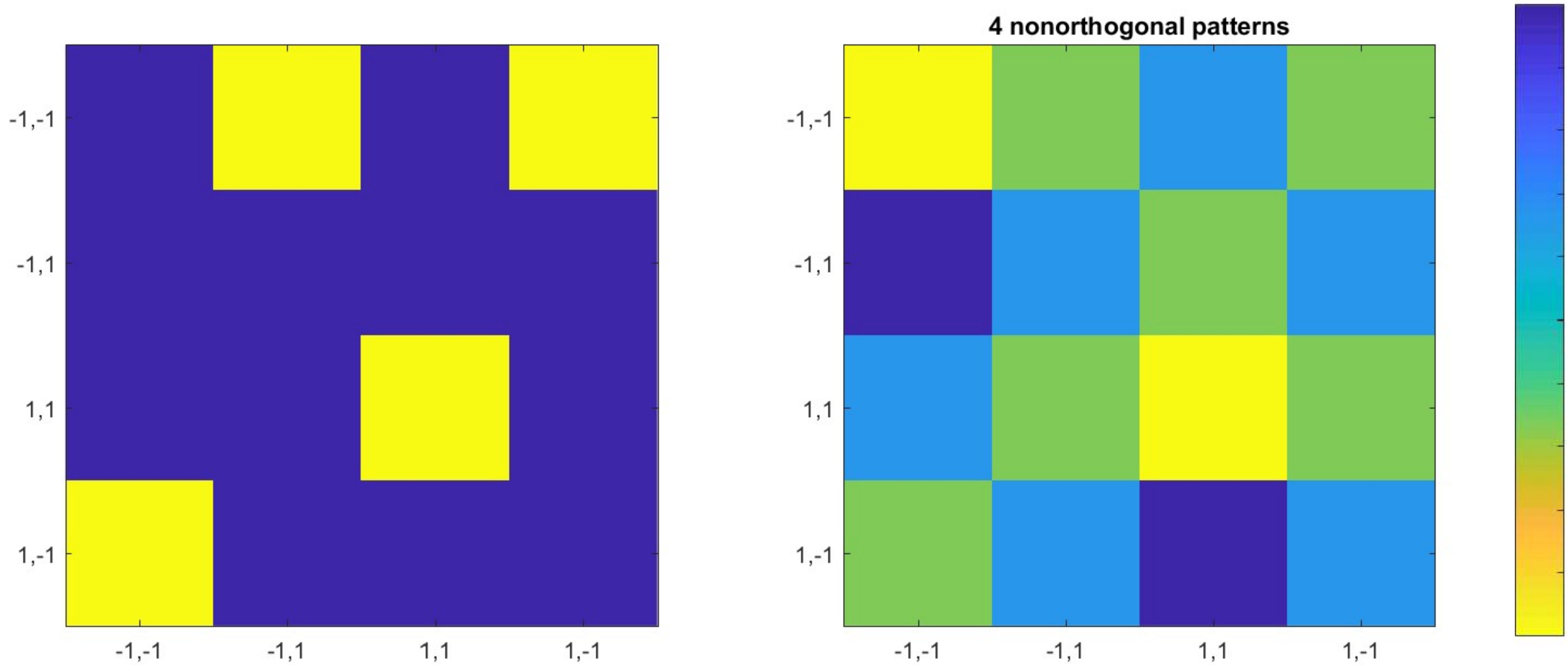
- Patterns in the corner are not recalled
  - They end up being attracted to the -1,-1 pattern
  - Note some “ghosts” ended up in the “well” of other patterns
    - So one of the patterns has stronger recall than the other two

Four orthogonal 4-bit patterns



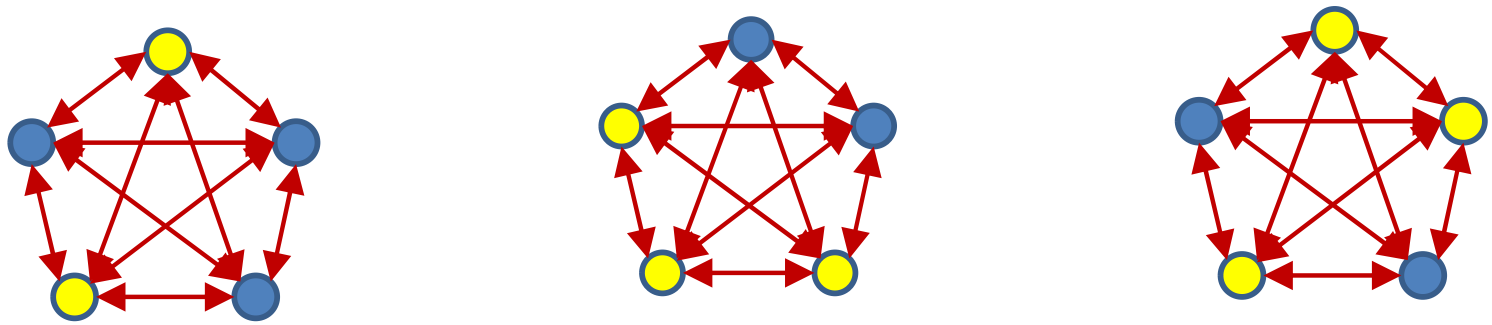
- All patterns are stationary, but none are stable
  - Total wipe out

*Four non*orthogonal 4-bit patterns



- One stable pattern
  - "Collisions" when the ghost of one pattern occurs next to another

How many patterns can we store?

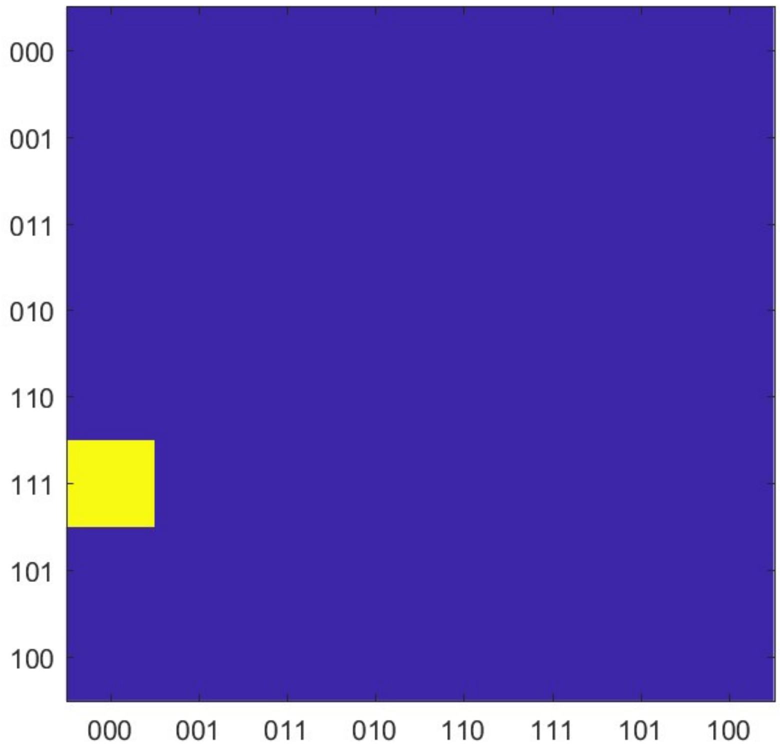


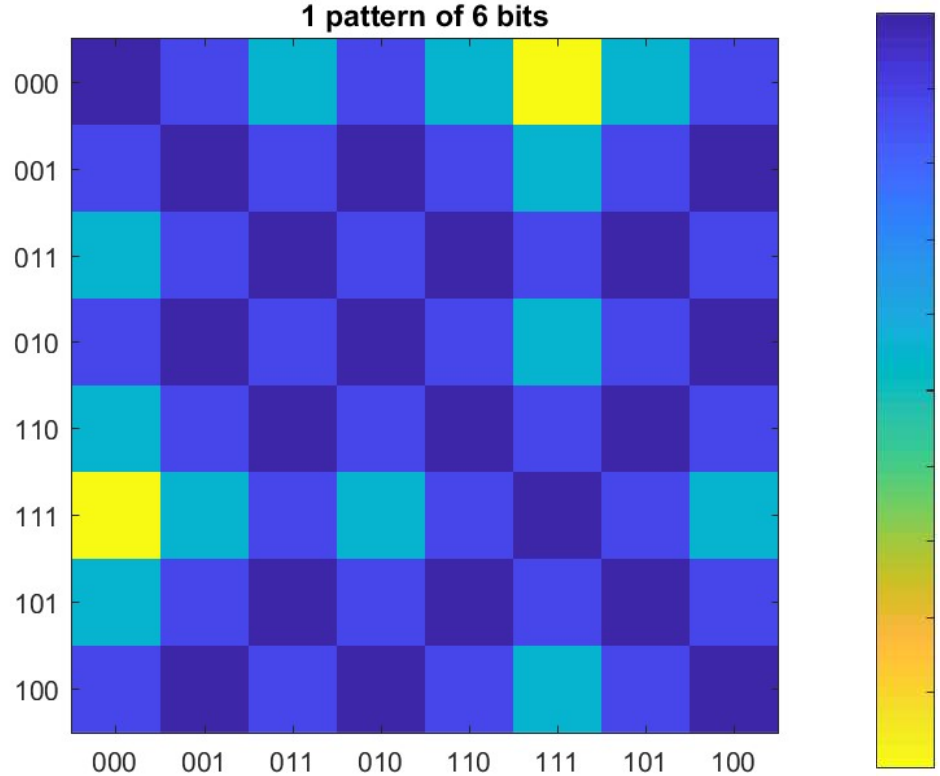
- Hopfield: For a network of N neurons can store up to $0.14N$ random patterns

- Apparently a fuzzy statement
  - What does it really mean to say "stores" 0.14N random patterns?
    - Stationary? Stable? No other local minima?
  - What if the patterns to store are not random?

- N=4 may not be a good case (N too small)

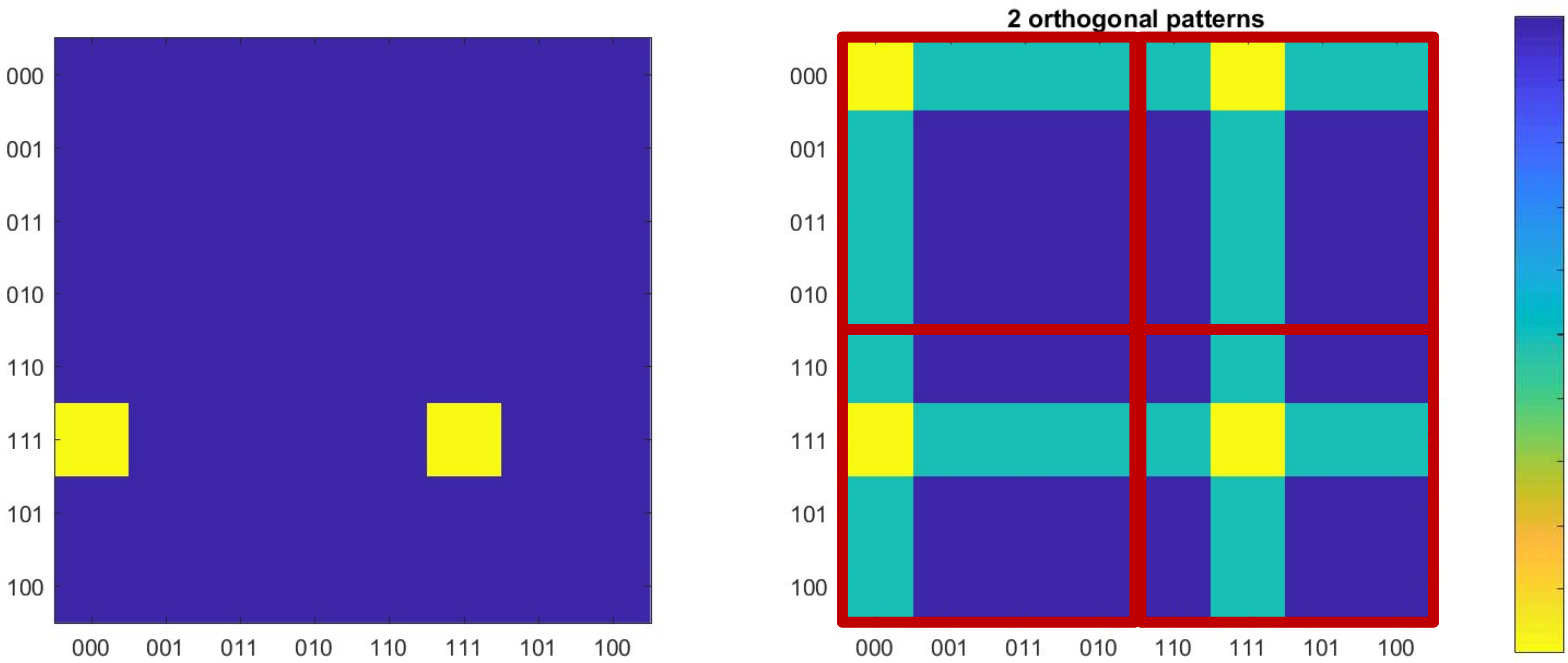
A 6-bit pattern





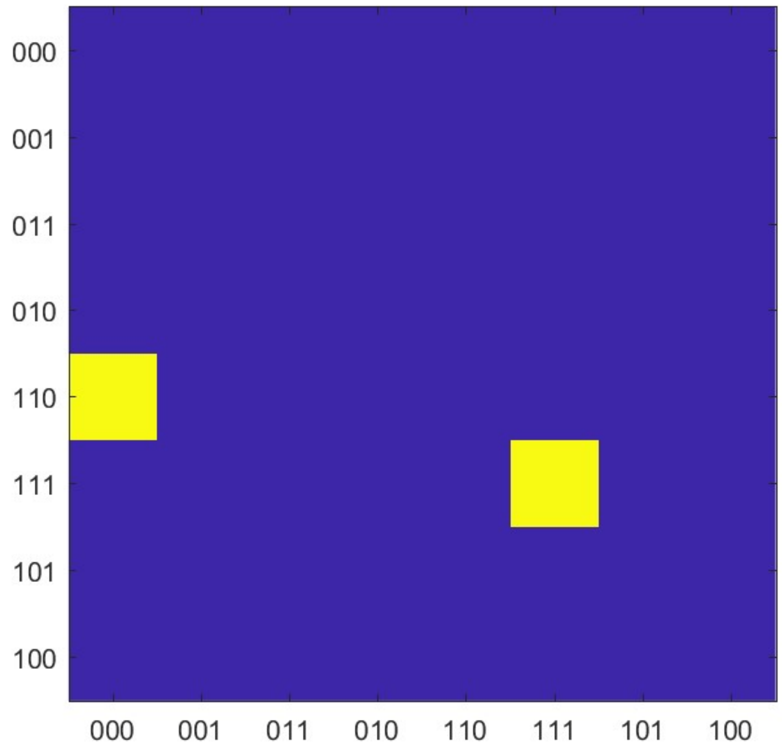
- Perfectly stationary and stable
- But many spurious local minima..
  – Which are "fake" memories

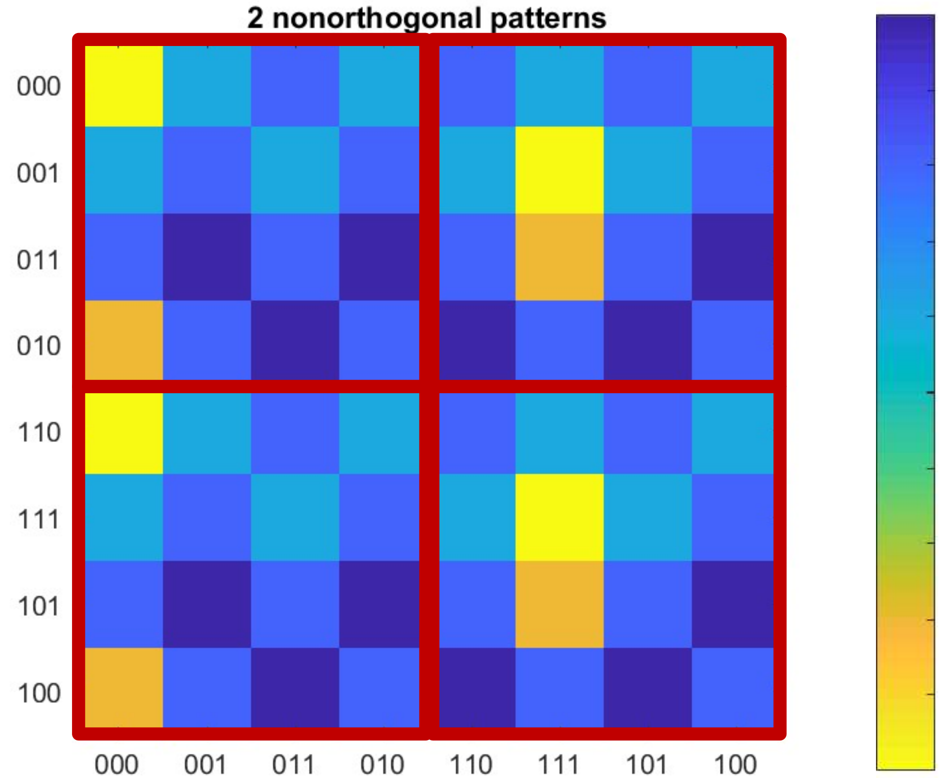
Two orthogonal 6-bit patterns



- Perfectly stationary and stable
- Several spurious "fake-memory" local minima..
  – Figure overstates the problem: actually a 3-D Kmap

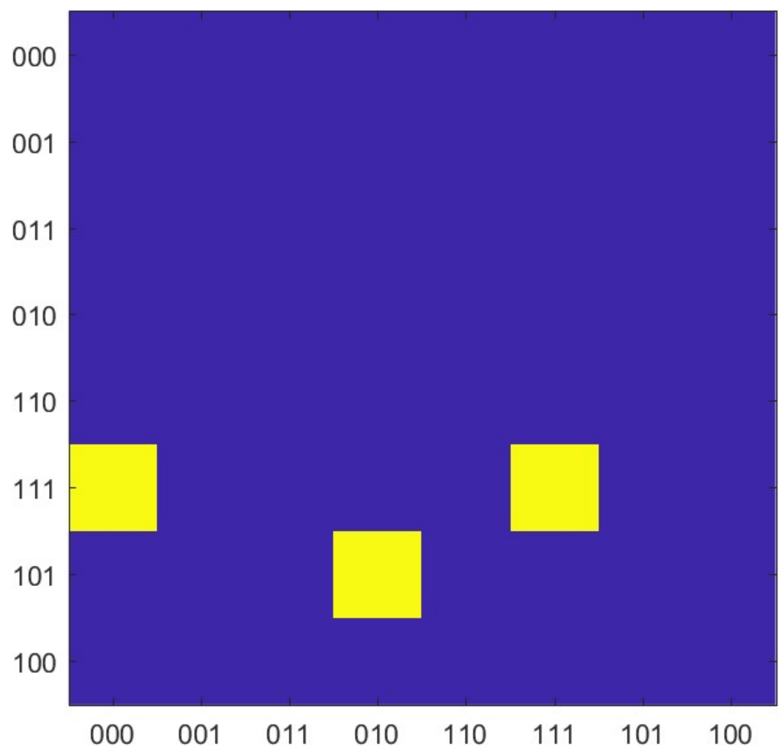
Two non-orthogonal 6-bit patterns

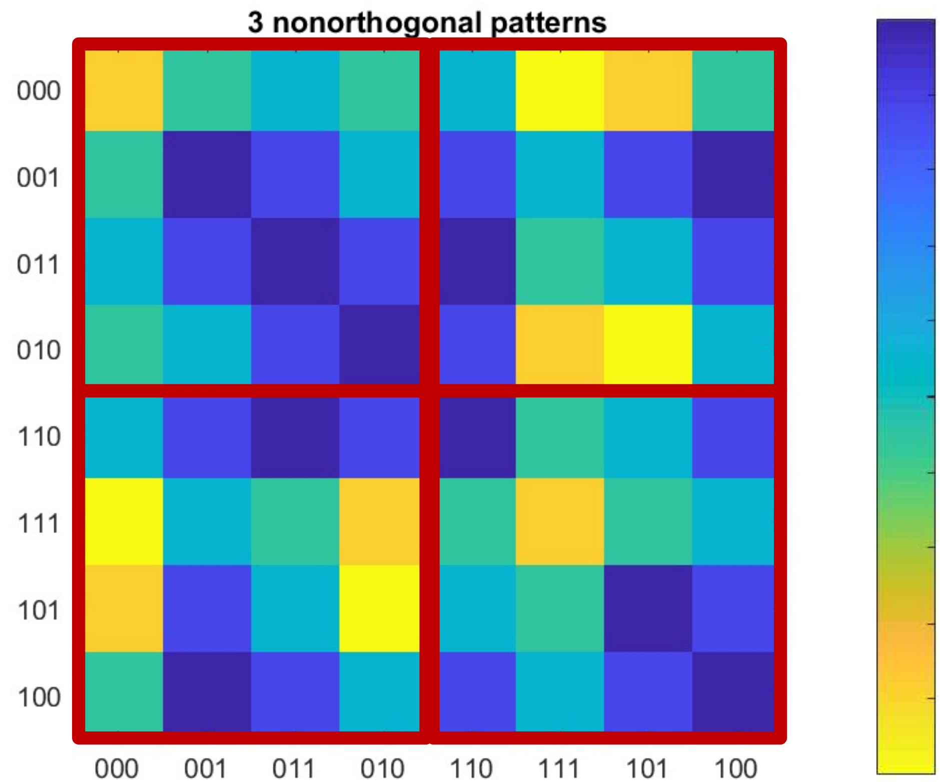




- Perfectly stationary and stable
- Some spurious “fake-memory” local minima..
  - But every stored pattern has “bowl”
  - *Fewer* spurious minima than for the orthogonal case

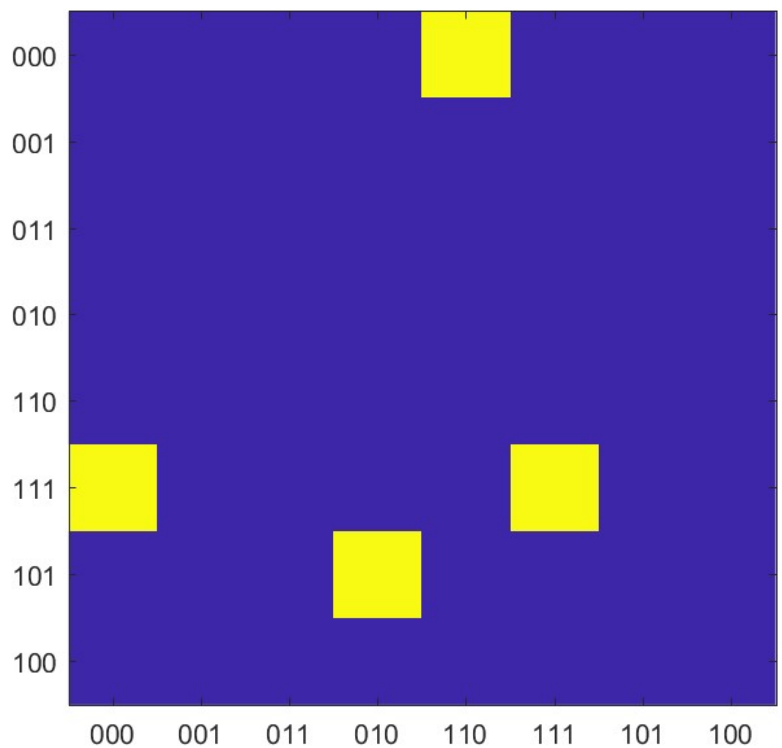
Three *non*-orthogonal 6-bit patterns

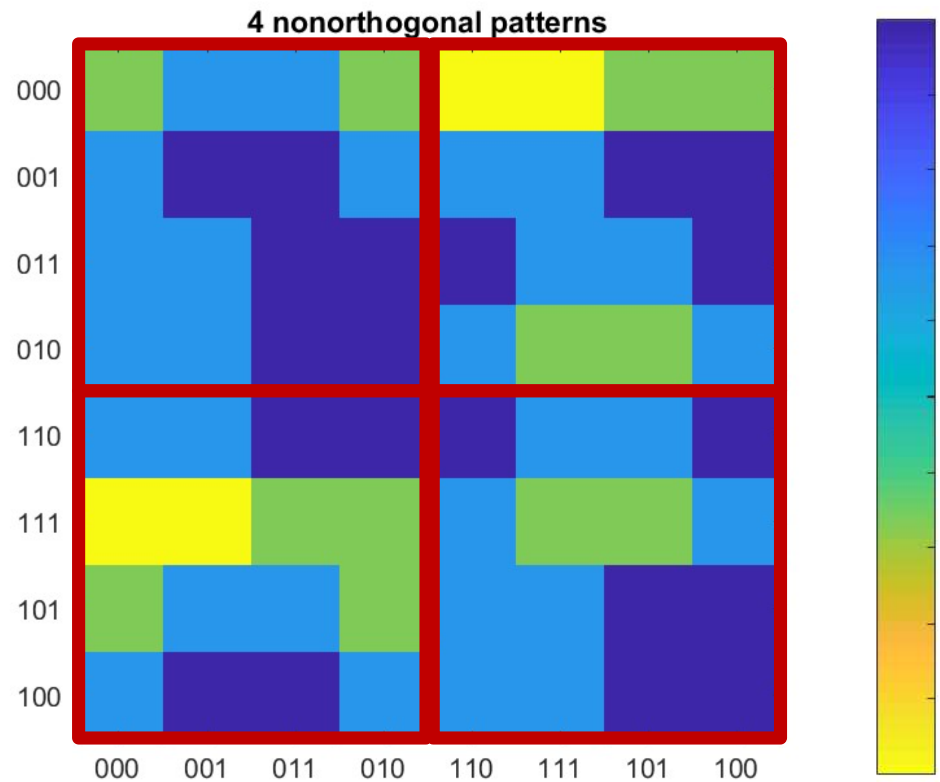




- Note: Cannot have 3 or more orthogonal 6-bit patterns..
- Patterns are perfectly stationary and stable (K > 0.14N)
- Some spurious “fake-memory” local minima..
  - But every stored pattern has “bowl”
  - *Fewer* spurious minima than for the orthogonal 2-pattern case

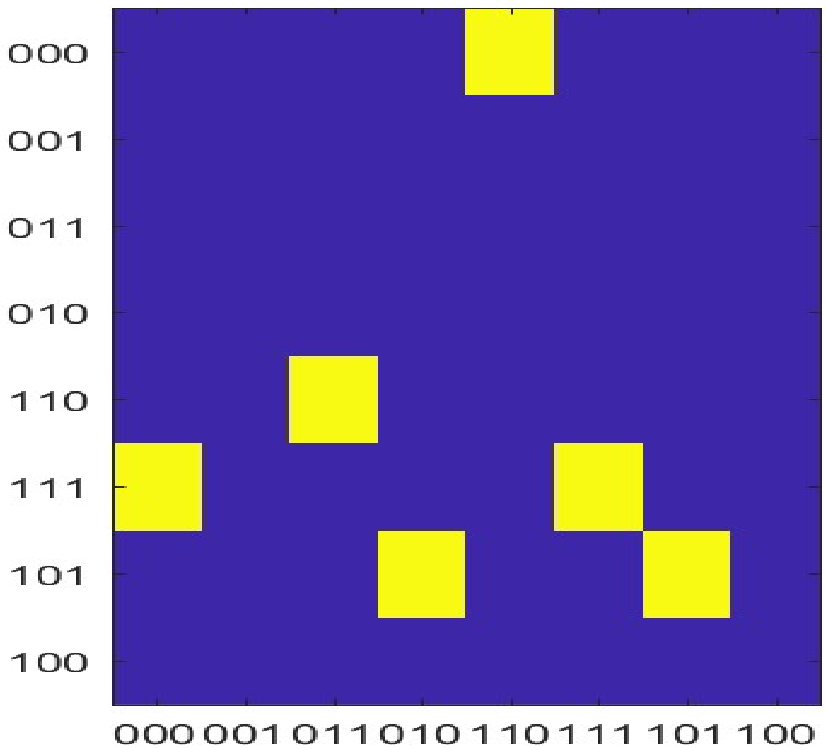
Four *non*-orthogonal 6-bit patterns

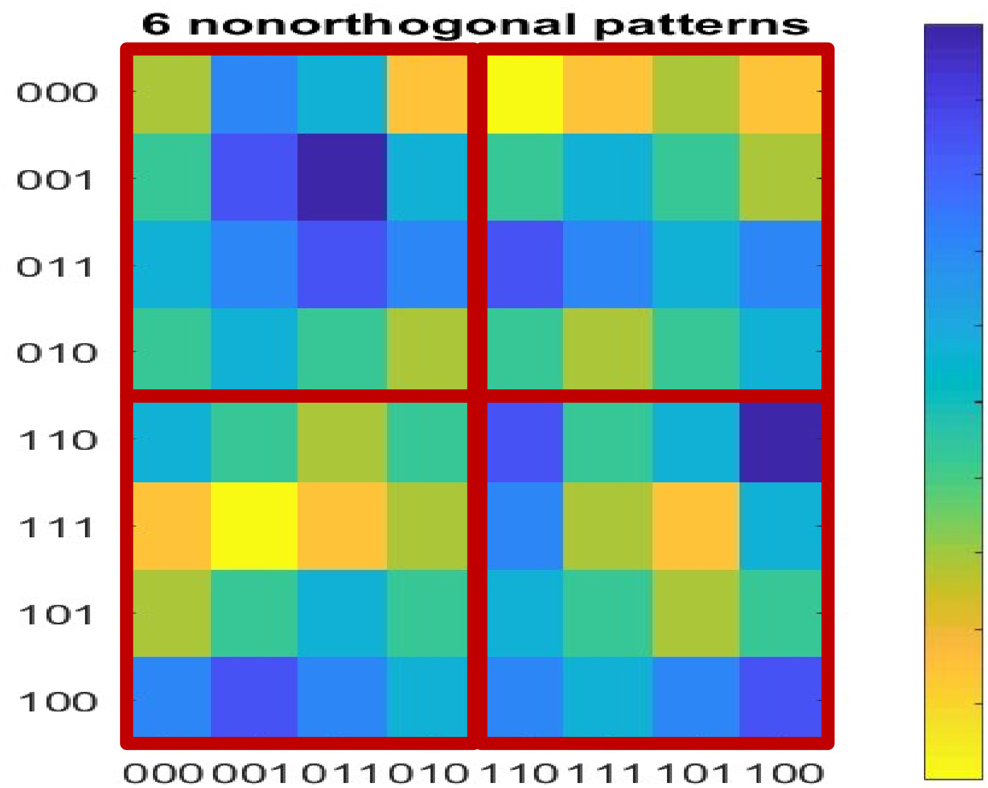




- Patterns are perfectly stationary for K > 0.14N
- *Fewer* spurious minima than for the orthogonal 2-pattern case
  - Most fake-looking memories are in fact ghosts..

Six *non*-orthogonal 6-bit patterns



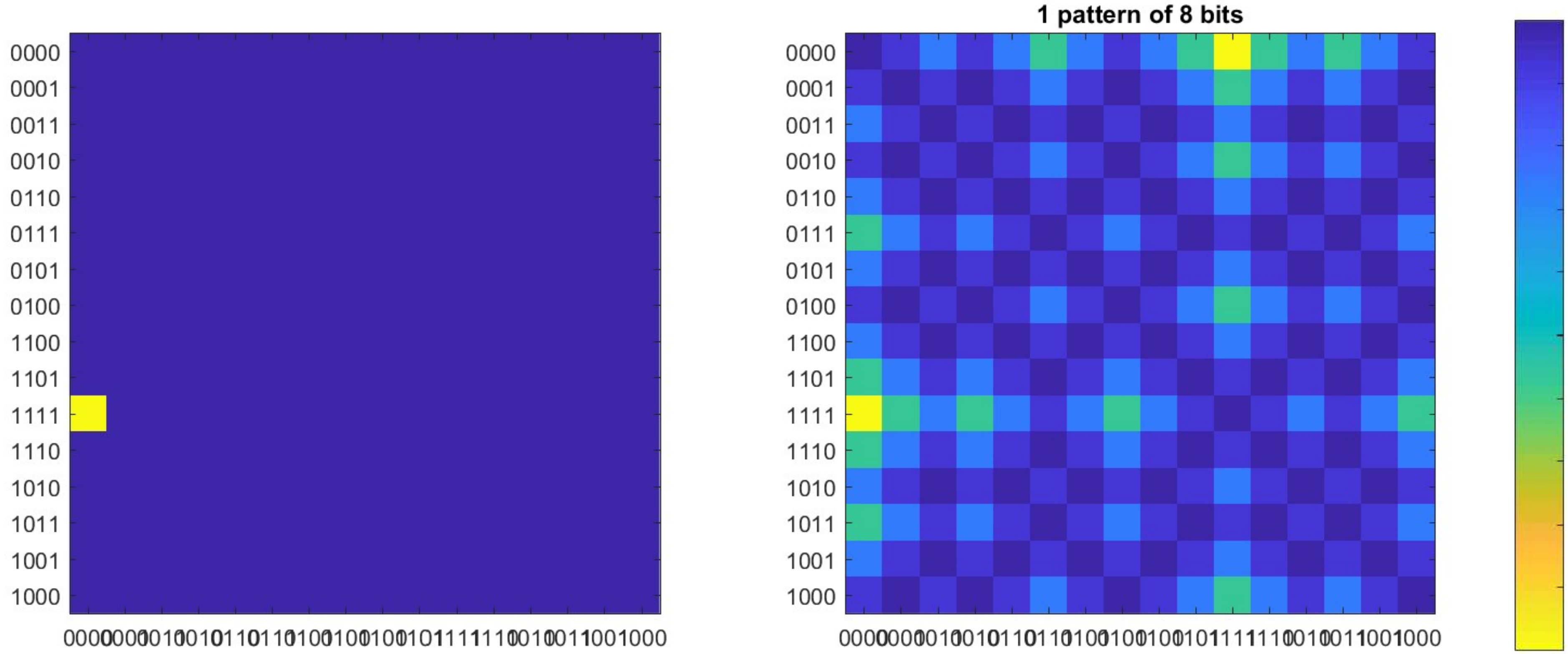


- Breakdown largely due to interference from “ghosts”
- But multiple patterns are stationary, and often stable
  - For K >> 0.14N

More visualization..

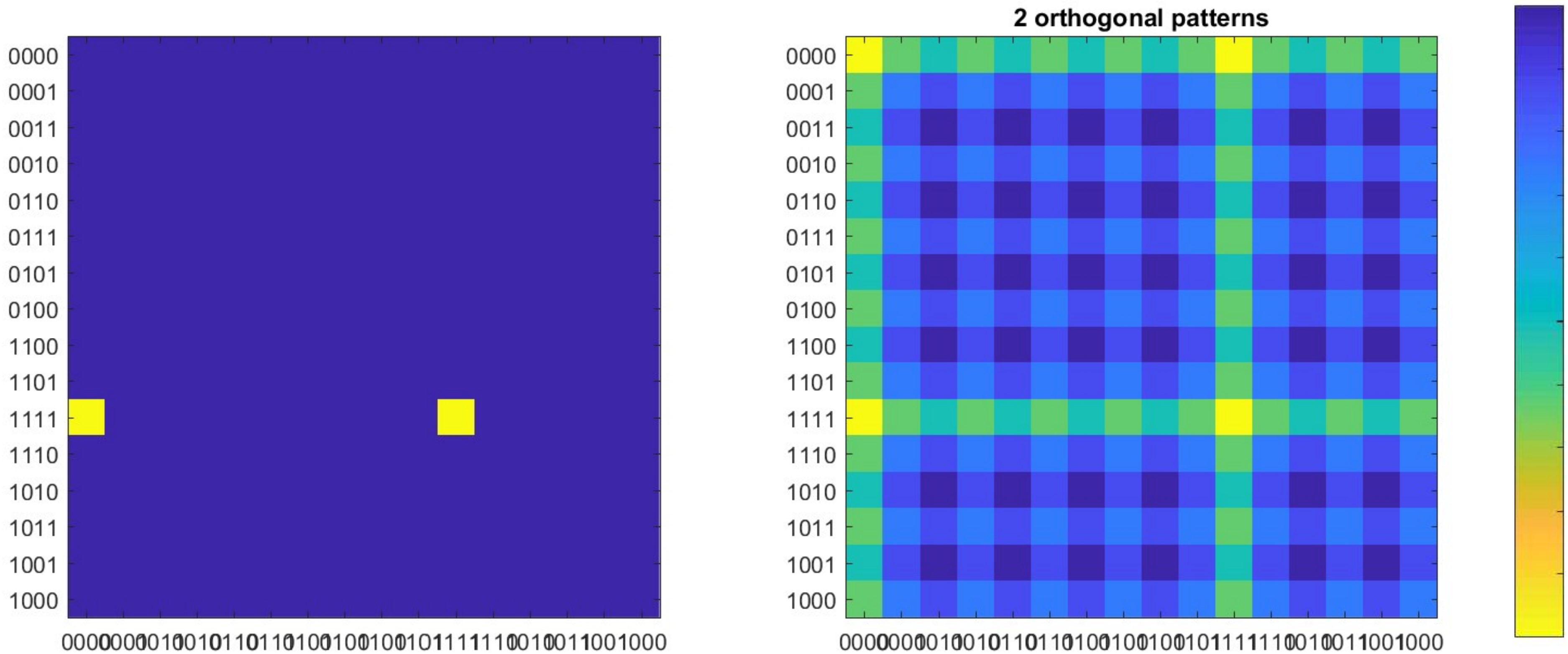
- Lets inspect a few 8-bit patterns
  - Keeping in mind that the Karnaugh map is now a 4-dimensional tesseract

One 8-bit pattern



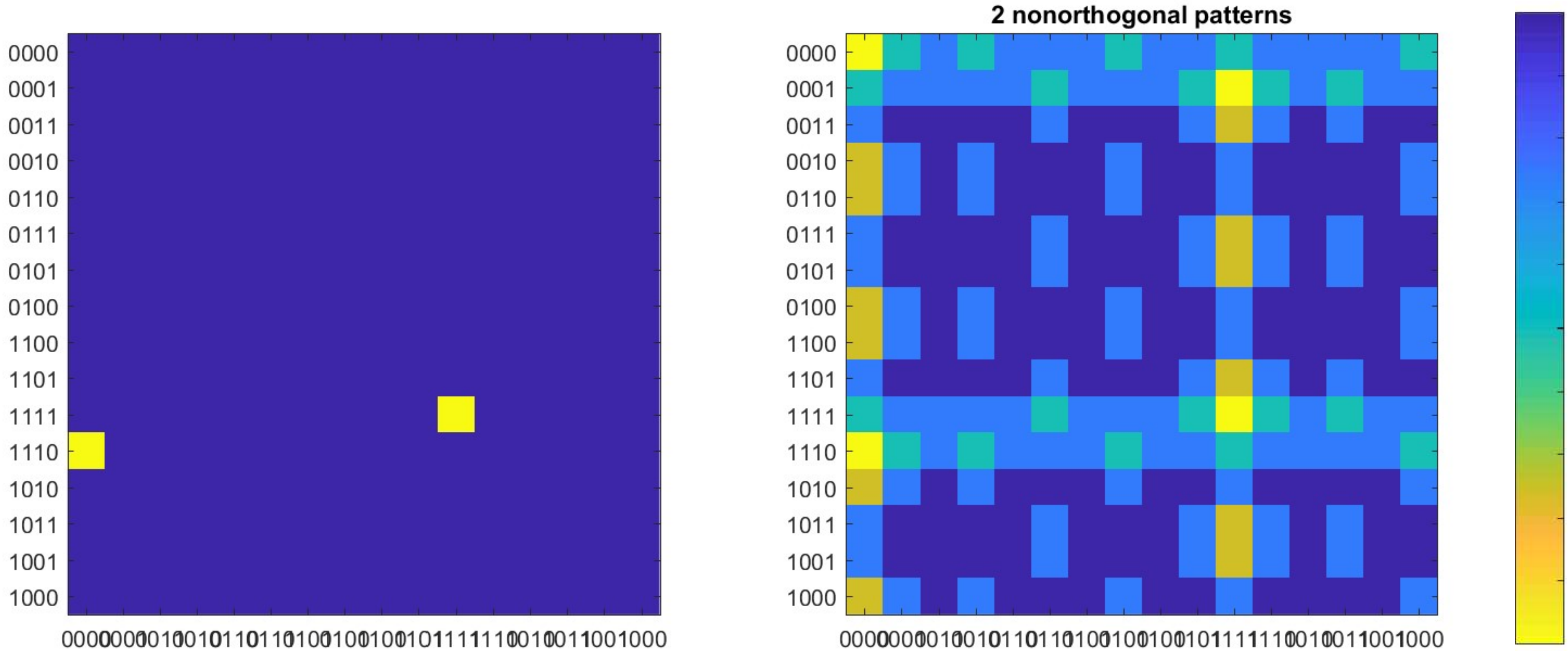
- Its actually cleanly stored, but there are a few spurious minima

Two orthogonal 8-bit patterns



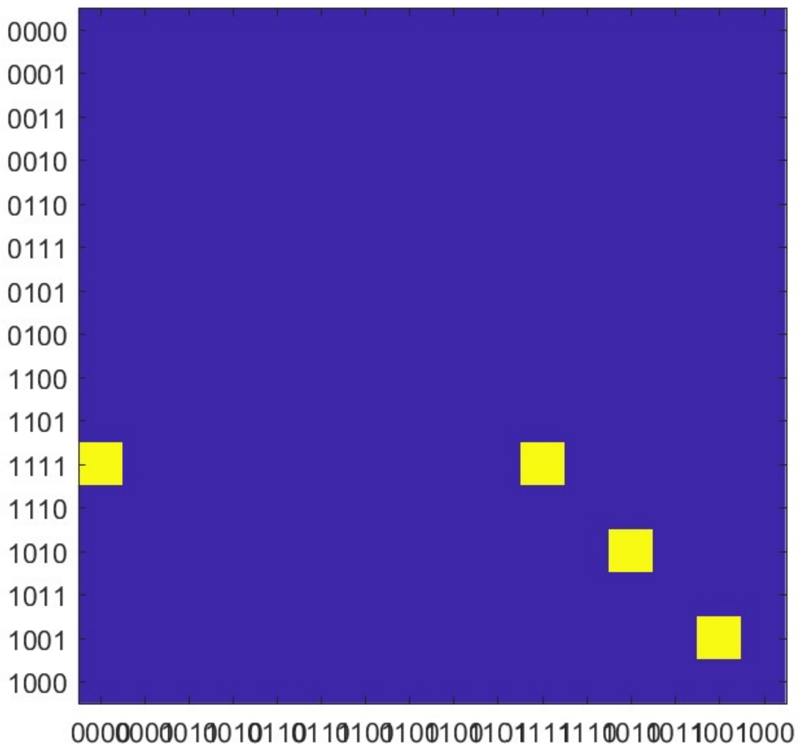
- Both have regions of attraction
- Some spurious minima

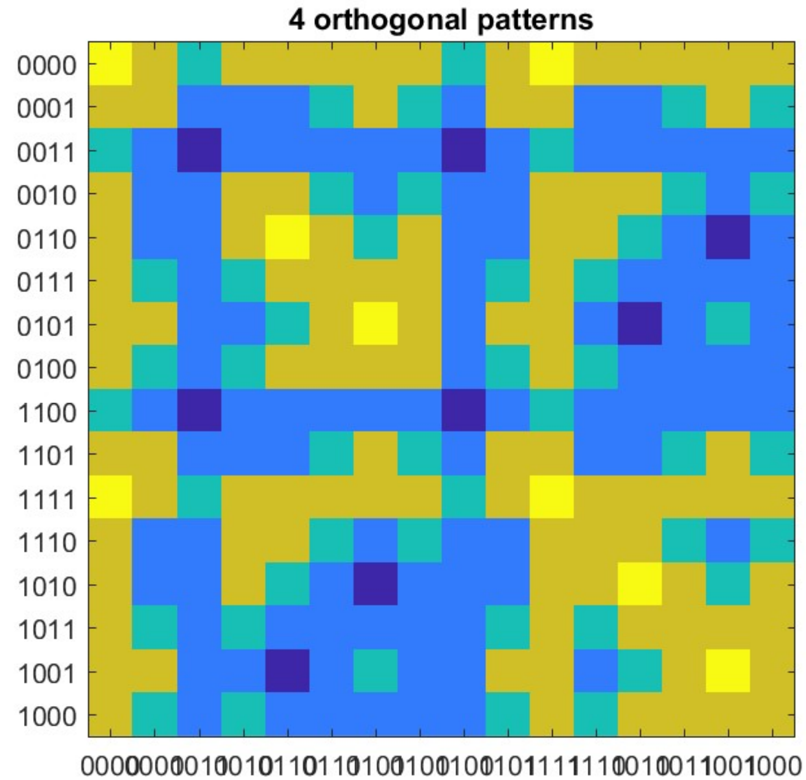
Two non-orthogonal 8-bit patterns



- Actually have fewer spurious minima
  - Not obvious from visualization..

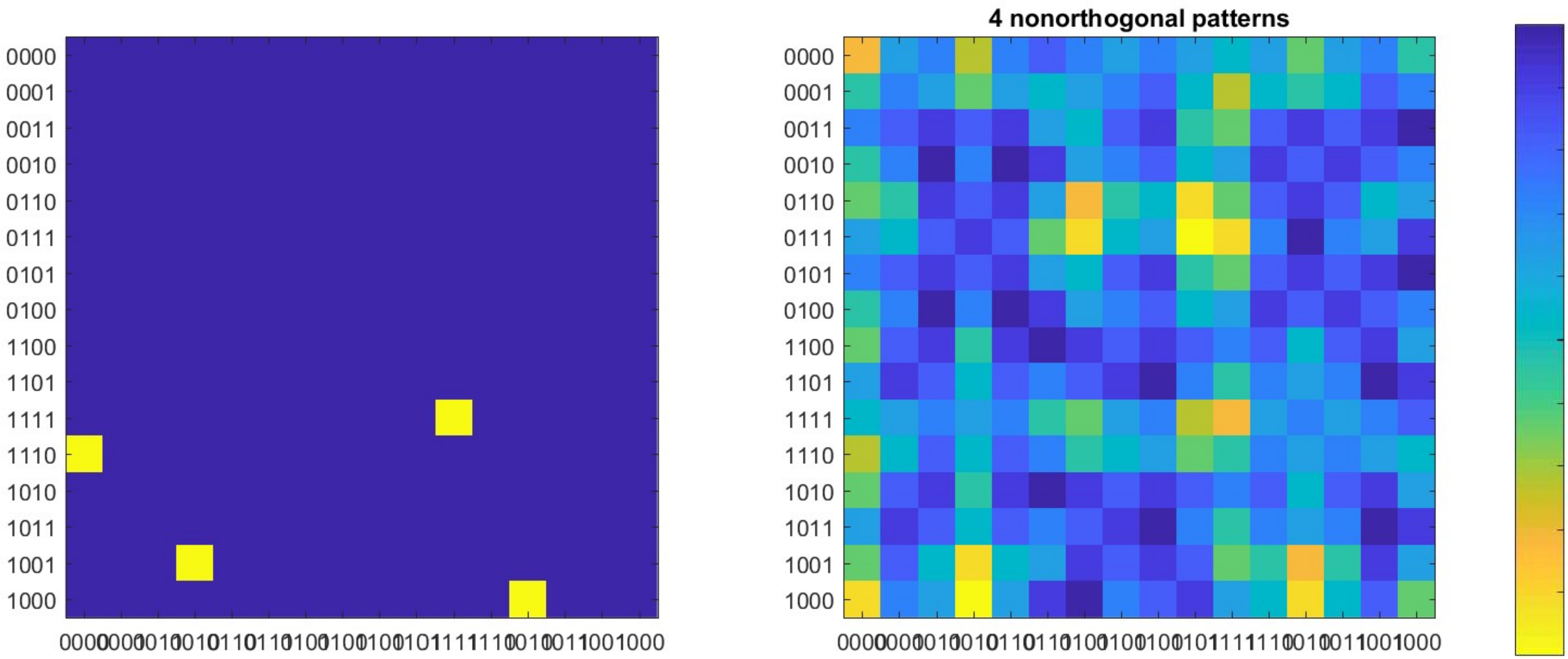
Four orthogonal 8-bit patterns





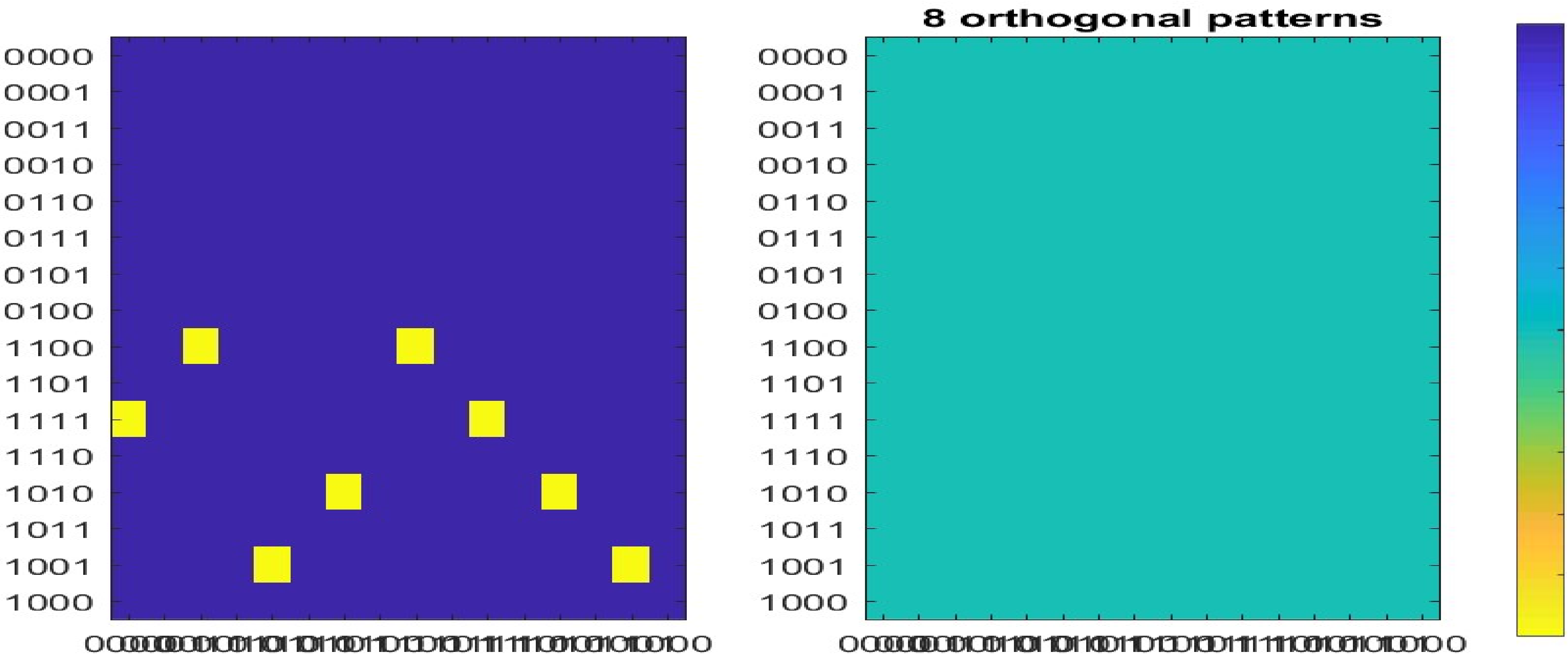
- Successfully stored

Four non-orthogonal 8-bit patterns



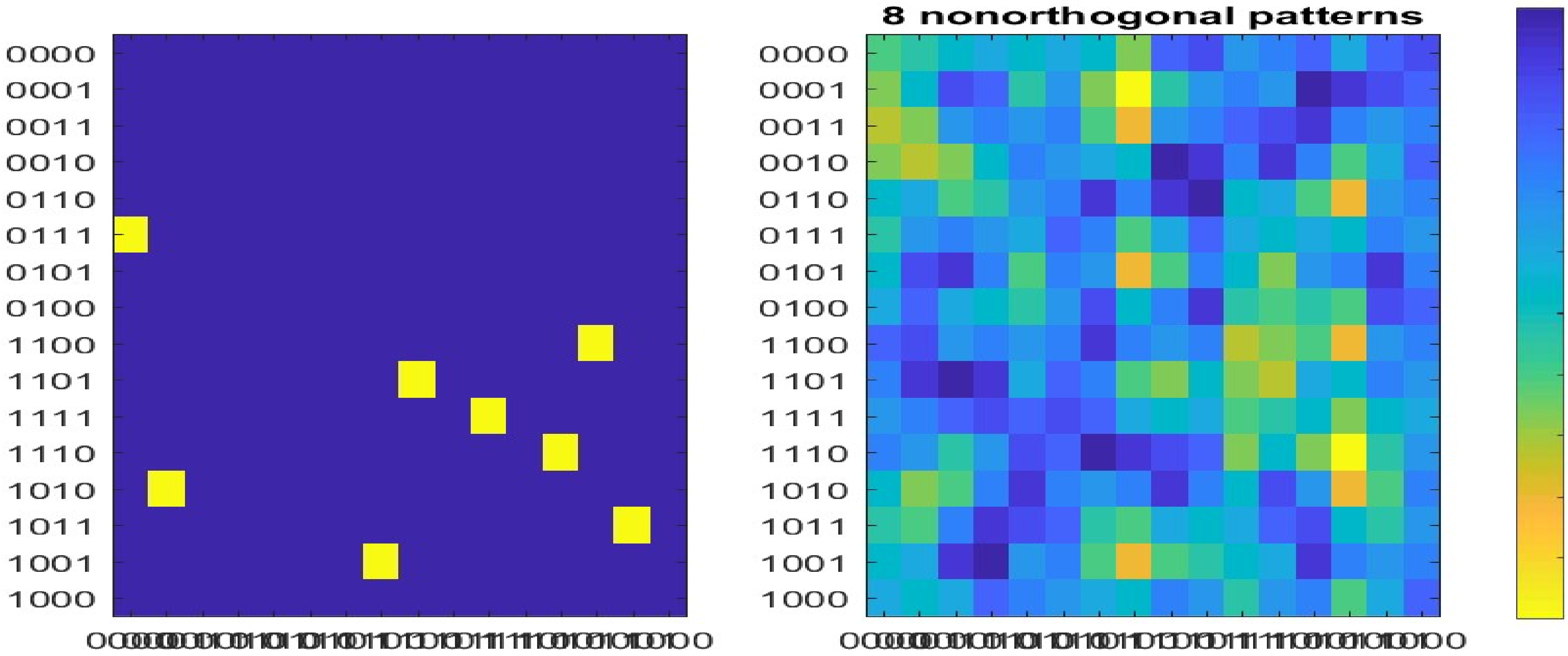
- Stored with interference from ghosts..

Eight orthogonal 8-bit patterns



- Wipeout

Eight non-orthogonal 8-bit patterns



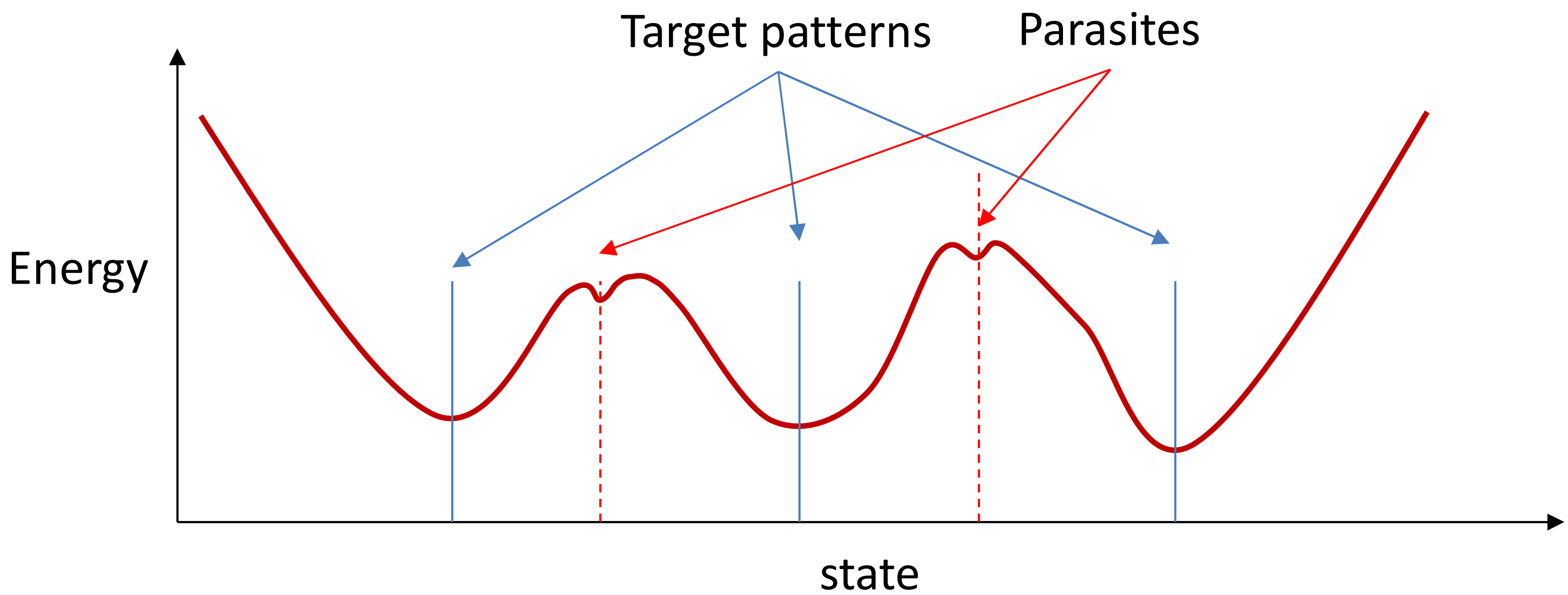
- Nothing stored
  - Neither stationary nor stable

Observations

- Many “parasitic” patterns
  – Undesired patterns that also become stable or attractors

- Apparently, a capacity to store *more* than 0.14N patterns

Parasitic Patterns



- Parasitic patterns can occur because sums of odd numbers of stored patterns are also stable for Hebbian learning:
  - $\mathbf{y}_{parasite} = sign(\mathbf{y}_a + \mathbf{y}_b + \mathbf{y}_c)$
- They are also from other random local energy minima from the weights matrices themselves

Capacity

- Seems possible to store K > 0.14N patterns
  - i.e. obtain a weight matrix W such that K > 0.14N patterns are stationary
  - Possible to make more than 0.14N patterns at-least 1-bit stable

- Patterns that are *non-orthogonal* easier to remember
  - I.e. patterns that are *closer* are easier to remember than patterns that are farther!!

- Can we attempt to get greater control on the process than Hebbian learning gives us?
  - Can we do *better* than Hebbian learning?
    - Better capacity and fewer spurious memories?

Story so far

- A Hopfield network is a loopy binary net with symmetric connections
  - Neurons try to align themselves to the local field caused by other neurons
- Given an initial configuration, the patterns of neurons in the net will evolve until the "energy" of the network achieves a local minimum
  - The network acts as a *content-addressable* memory
    - Given a damaged memory, it can evolve to recall the memory fully
- The network must be designed to store the desired memories
  - Memory patterns must be *stationary* and *stable* on the energy contour
- Network memory can be trained by Hebbian learning
  - Guarantees that a network of N bits trained via Hebbian learning can store 0.14N random patterns with less than 0.4% probability that they will be unstable
- However, empirically it appears that we may sometimes be able to store *more than* 0.14N patterns

Poll 3: @1798

Mark all that are true

- We can try to "assign" memories to a Hopfield network through Hebbian learning of the weights matrix
- All patterns learned through Hebbian learning will be "remembered"
- The N-bit Hopfield network has the capacity to remember up to 0.14N patterns

Poll 3

Mark all that are true

- **We can try to "assign" memories to a Hopfield network through Hebbian learning of the weights matrix**
- All patterns learned through Hebbian learning will be "remembered"
- The N-bit Hopfield network has the capacity to remember up to 0.14N patterns

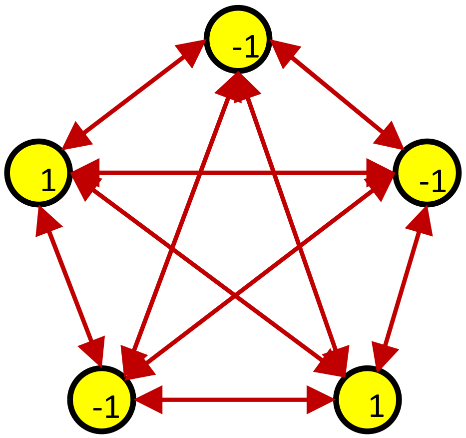
Bold Claim

- I can *always* store (upto) N orthogonal patterns such that they are stationary!

  – Why?

- I can avoid spurious memories by adding some noise during recall!

Recap: Hebbian Learning to Store a Specific Pattern



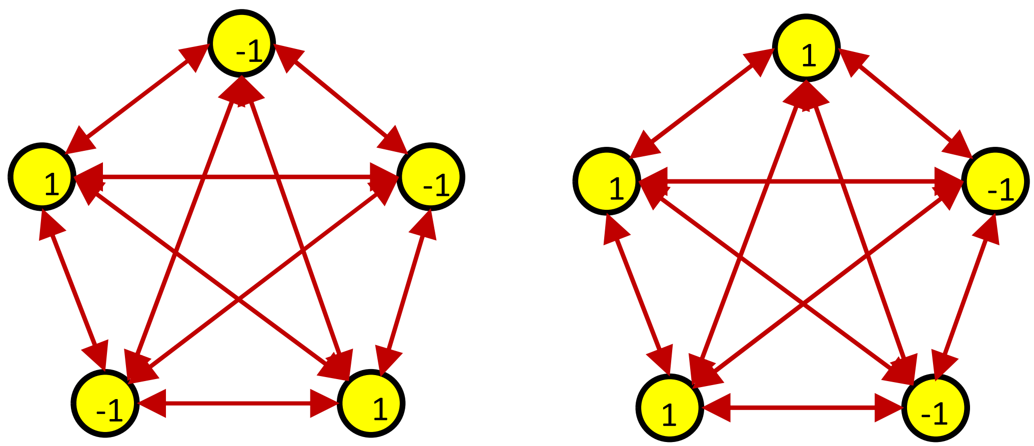
HEBBIAN LEARNING:

$$w_{ji} = y_j y_i$$

$$\mathbf{W} = \mathbf{y}_p \mathbf{y}_p^T - \mathrm{I}$$

- For a single stored pattern, Hebbian learning results in a network for which the target pattern is a global minimum

Storing multiple patterns



- Let $\mathbf{y}_p$ be the vector representing p-th pattern
- Let $\mathbf{Y} = [\mathbf{y}_1\ \mathbf{y}_2\ \ldots]$ be a matrix with all the stored patterns
- Then..

$$\mathbf{W} = \frac{1}{N_p}\sum_{\boldsymbol{p}}(\mathbf{y}_p\mathbf{y}_p^T - \mathrm{I}) = \frac{1}{N_p}\mathbf{Y}\mathbf{Y}^T - \mathbf{I}$$

Number of patterns

W is a positive semi-definite matrix

A minor adjustment

- Note behavior of $\mathbf{E}(\mathbf{y}) = \mathbf{y}^T\mathbf{W}\mathbf{y}$ with

$$\mathbf{W} = \frac{1}{N_p}\mathbf{Y}\mathbf{Y}^T - \mathbf{I}$$

- Is identical to behavior with

$$\mathbf{W} = \mathbf{Y}\mathbf{Y}^T$$

- Since

$$\mathbf{y}^T\left(\frac{1}{N_p}\mathbf{Y}\mathbf{Y}^T - \mathbf{I}\right)\mathbf{y} = \frac{1}{N_p}\mathbf{y}^T\mathbf{Y}\mathbf{Y}^T\mathbf{y} - N$$

- But $\mathbf{W} = \mathbf{Y}\mathbf{Y}^T$ is easier to analyze. Hence in the following slides we will use $\mathbf{W} = \mathbf{Y}\mathbf{Y}^T$

Energy landscape only differs by an additive constant and a scaling

Location of minima remain same

A minor adjustment

- Note behavior of $\mathbf{E}(\mathbf{y}) = \mathbf{y}^T\mathbf{W}\mathbf{y}$ with

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Both have the same Eigen vectors

vior with

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A minor adjustment

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Both have the
same Eigen vectors

vior with

$$\mathbf{W} = \mathbf{Y}\mathbf{Y}^T$$

Energy landscape
only differs by
an additive constant and
a scaling

Location
of minima remain same

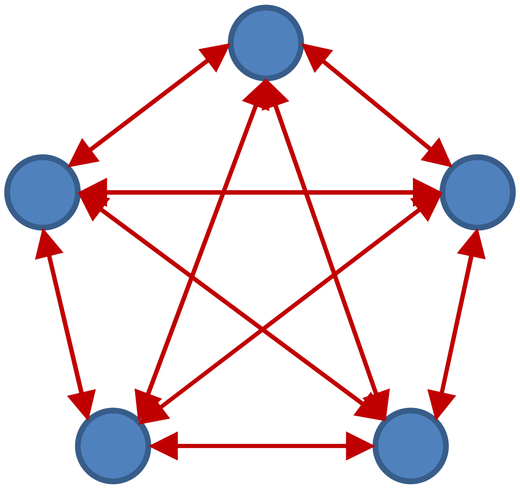
- Si

NOTE: This
is a positive
semidefinite matrix

$$' - \mathbf{I} \Big)\mathbf{y} = \frac{1}{N_p}\mathbf{y}^T\mathbf{Y}\mathbf{Y}^T\mathbf{y} - N$$

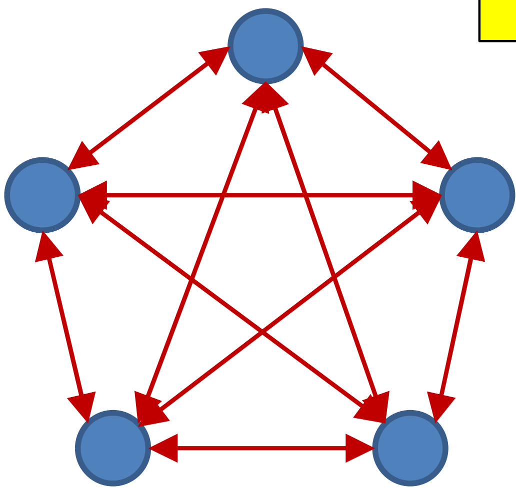
- But $\mathbf{W} = \mathbf{Y}\mathbf{Y}^T$ is easier to analyze. Hence in the following slides we will use $\mathbf{W} = \mathbf{Y}\mathbf{Y}^T$

Consider the energy function



$$E = -\frac{1}{2}\mathbf{y}^T\mathbf{W}\mathbf{y}$$

Consider the energy function



This is a quadratic!

For Hebbian learning
W is positive semidefinite

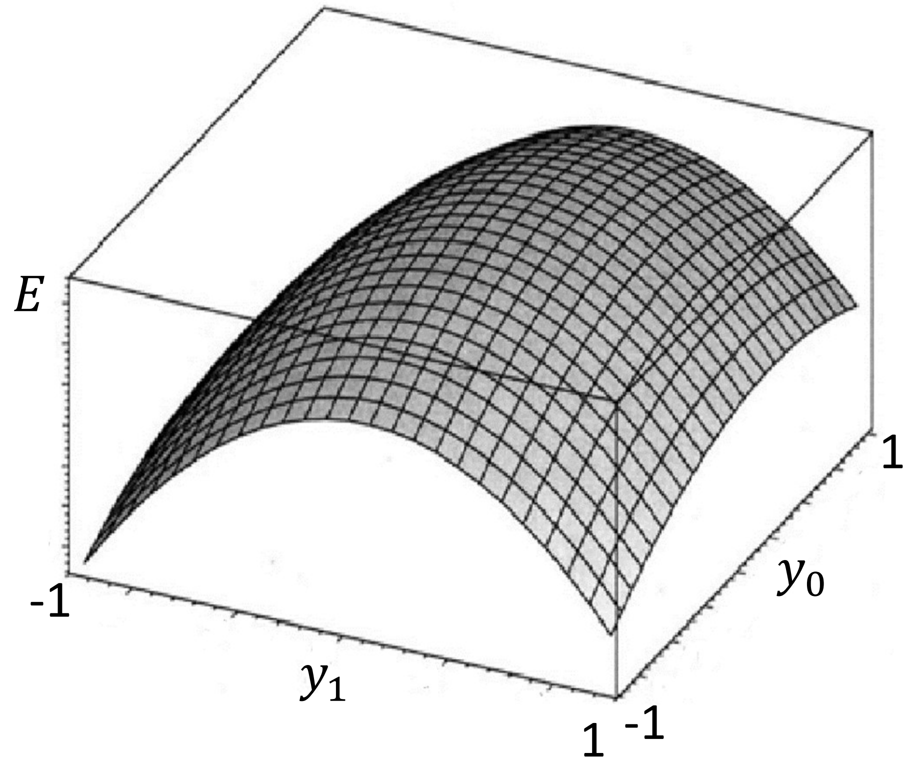
E is concave

$$E = -\frac{1}{2}\mathbf{y}^T\mathbf{W}\mathbf{y}$$

- The Energy function is concave if $\mathbf{W}$ is positive (semi) definite

The Energy function

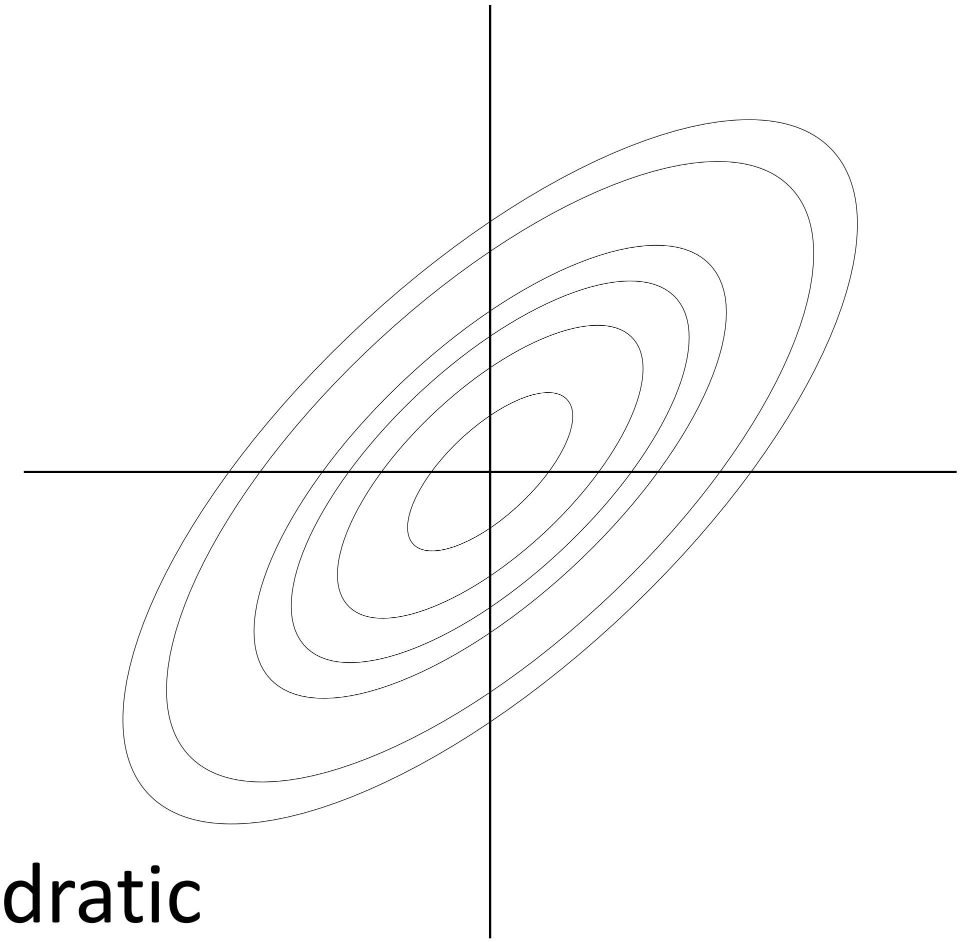
$$E = -\frac{1}{2}\mathbf{y}^T\mathbf{W}\mathbf{y}$$



- E is a concave quadratic

The Energy function

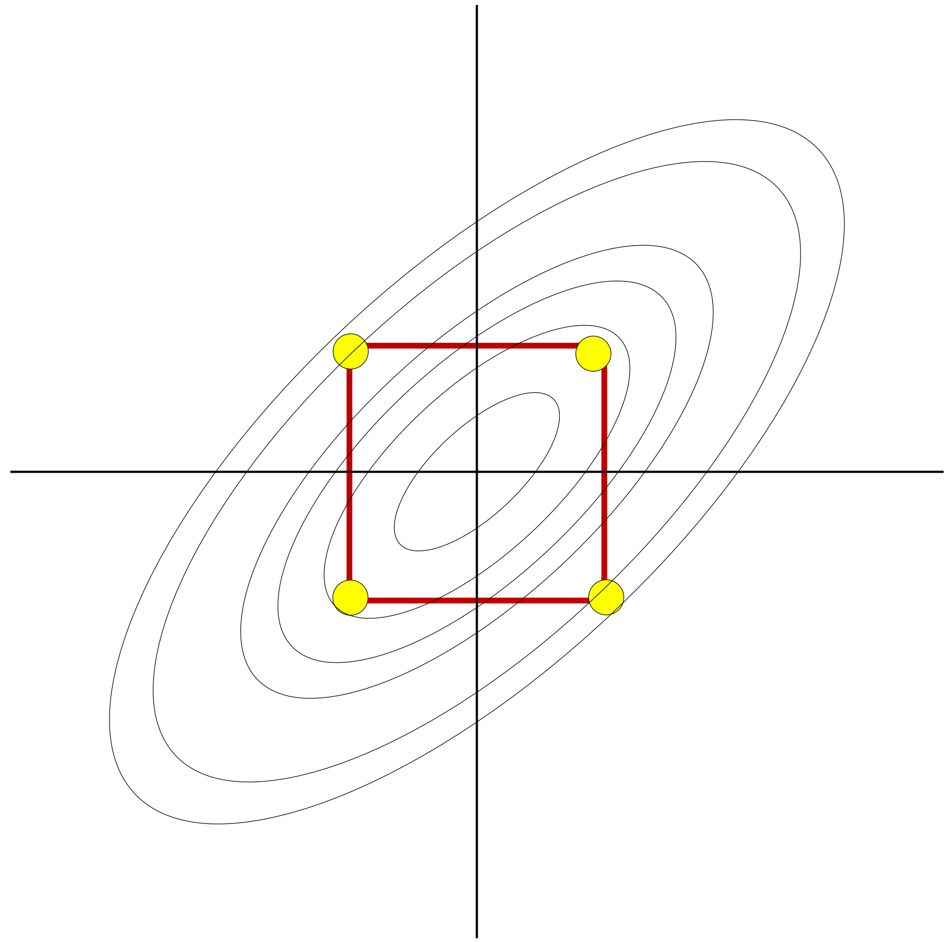
$$E = -\frac{1}{2}\mathbf{y}^T\mathbf{W}\mathbf{y}$$



- E is a concave quadratic
  - Shown from above (assuming 0 bias)

The energy function

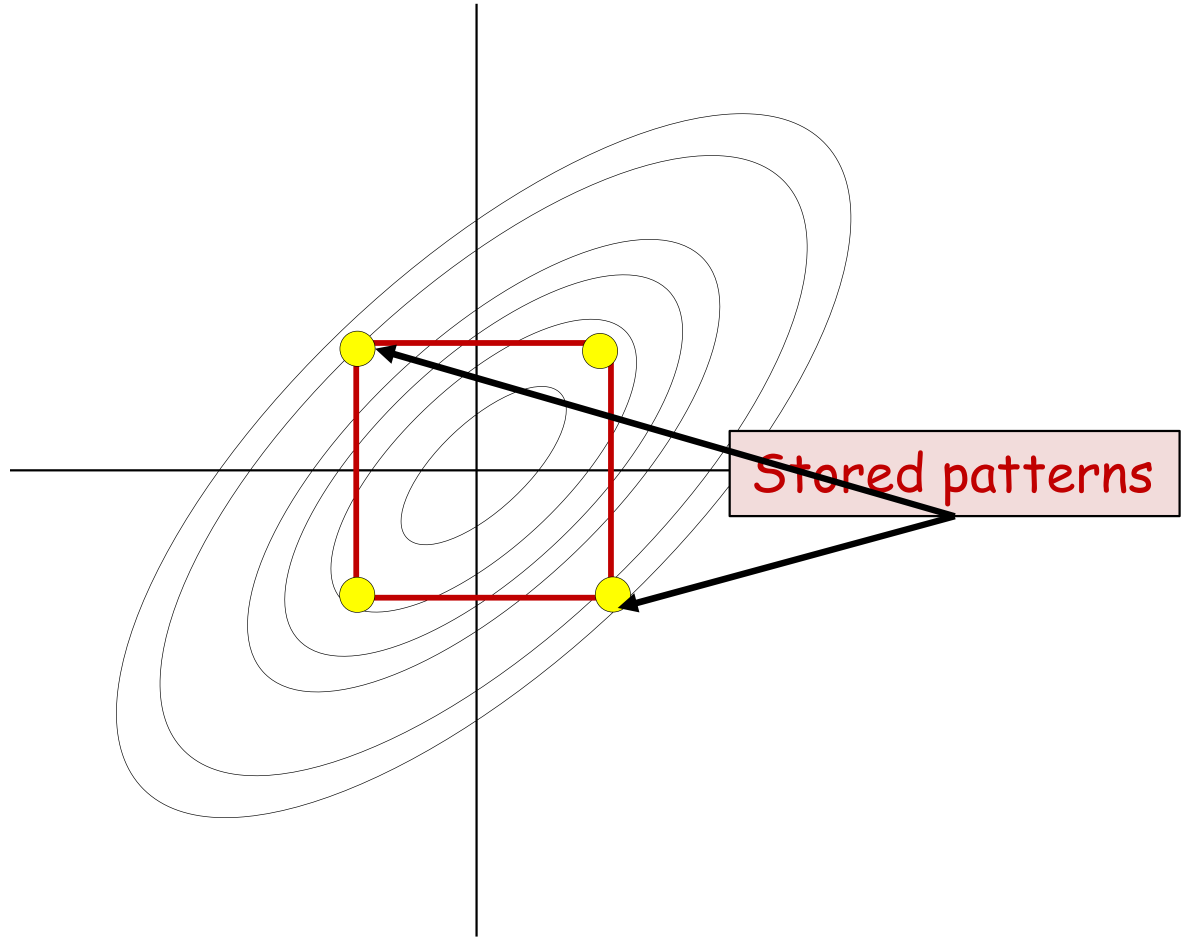
$$E = -\frac{1}{2}\mathbf{y}^T\mathbf{W}\mathbf{y}$$



- E is a concave quadratic
  - Shown from above (assuming 0 bias)
- The minima will lie on the boundaries of the hypercube
  - But components of y can only take values ± 1
  - I.e. y lies on the corners of the unit hypercube

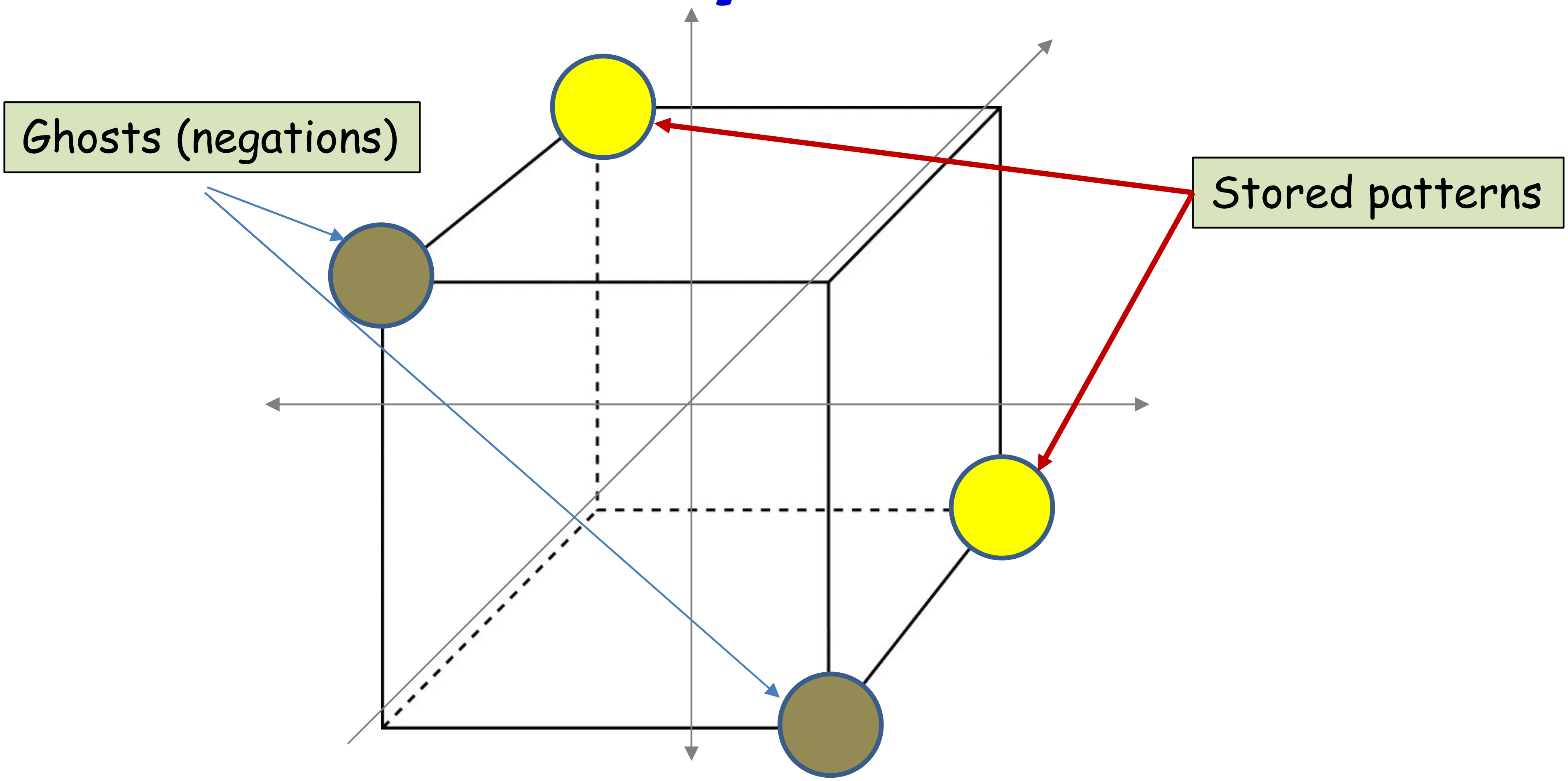
The energy function

$$E = -\frac{1}{2}\mathbf{y}^T\mathbf{W}\mathbf{y}$$



- The stored values of $\mathbf{y}$ are the ones where all adjacent corners are lower on the quadratic

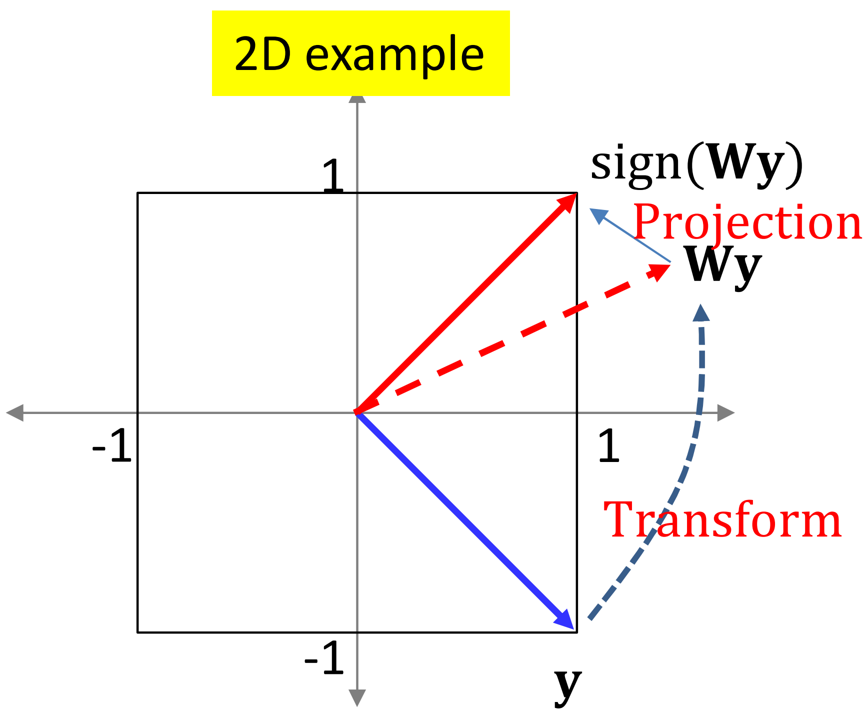
Patterns you can store

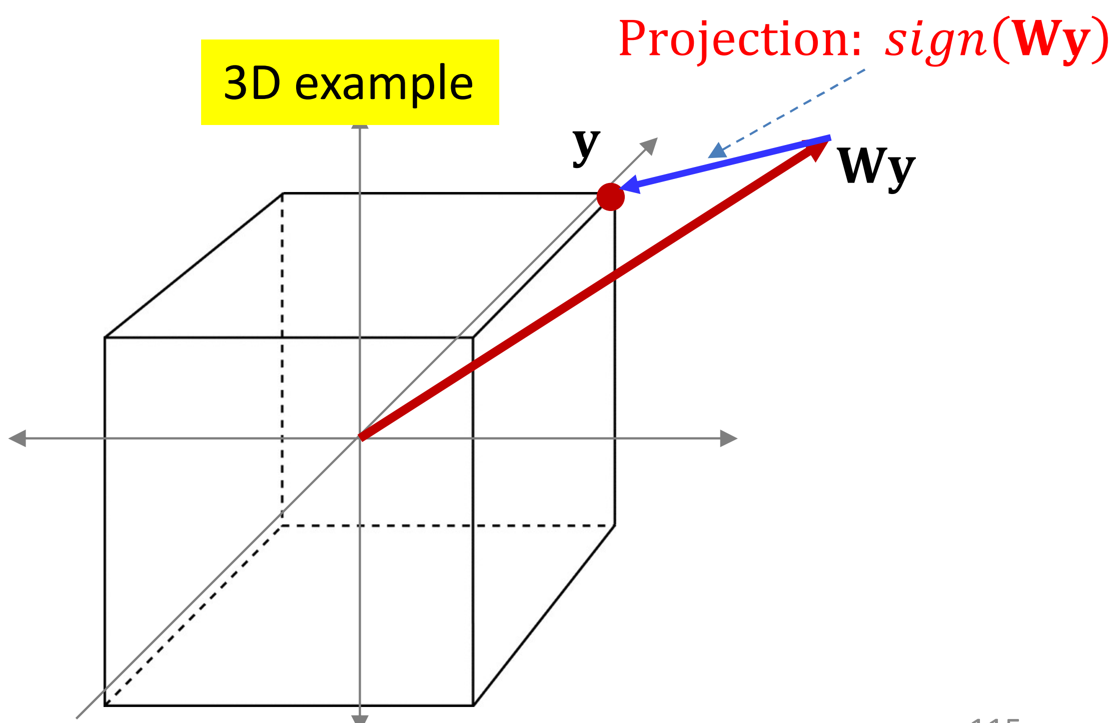


- All patterns are on the corners of a hypercube
  - If a pattern is stored, it's "ghost" is stored as well
  - Intuitively, patterns must ideally be maximally far apart
    - Though this doesn't seem to hold for Hebbian learning

Evolution of the network

- Note: for real vectors $sign(\mathbf{y})$ is a projection
  - Projects $\mathbf{y}$ onto the nearest corner of the hypercube
  - It "quantizes" the space into orthants
- Response to field: $\mathbf{y} \leftarrow sign(\mathbf{W}\mathbf{y})$
  - Each step rotates the vector $\mathbf{y}$ and then projects it onto the nearest corner





Storing patterns

- A pattern $\mathbf{y}_P$ is stored if:
  - $sign(\mathbf{W}\mathbf{y}_p) = \mathbf{y}_p$ for all target patterns
- Training: Design $\mathbf{W}$ such that this holds
- Simple solution: $\mathbf{y}_p$ is an Eigenvector of $\mathbf{W}$
  - And the corresponding Eigenvalue is positive

$$\mathbf{W}\mathbf{y}_p = \lambda\mathbf{y}_p$$

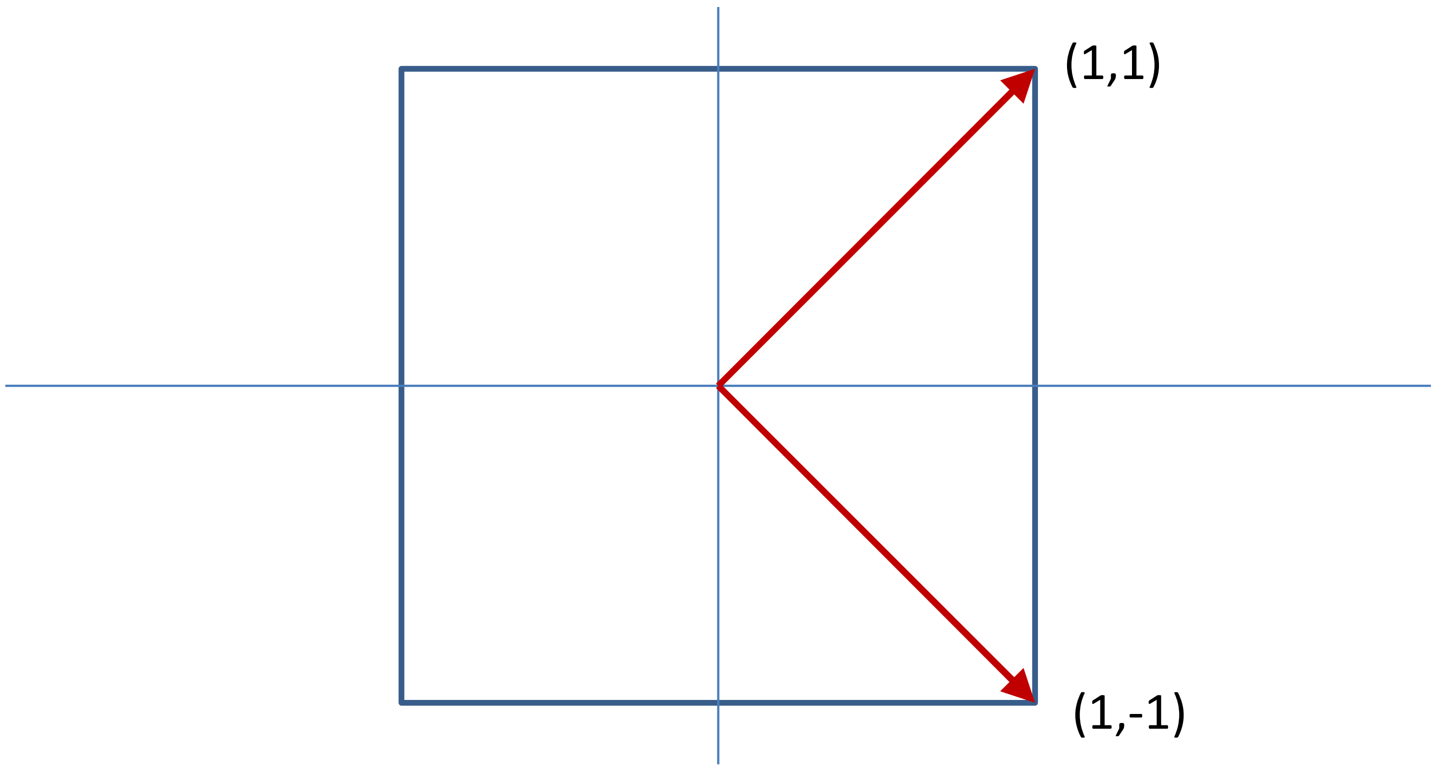
  - More generally orthant($\mathbf{W}\mathbf{y}_p$) = orthant($\mathbf{y}_p$)
- How many such $\mathbf{y}_p$ can we have?

Random fact that should interest you

- Number of ways of selecting two N-bit binary patterns $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ such that they differ from one another in exactly $N/2$ bits is $\mathcal{O}\left(2^{\frac{3N}{2}}\right)$

- The size of the largest set of N-bit binary patterns $\{\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots\}$ that *all* differ from one another in exactly $N/2$ bits is at most N
  - Trivial proof.. ☺

Only N patterns?



- Symmetric weight matrices have orthogonal Eigen vectors
- You can have max N orthogonal vectors in an N-dimensional space

random fact that should interest you

- The Eigenvectors of any symmetric matrix $\mathbf{W}$ are orthogonal

- The Eigen*values* may be positive or negative

Storing more than one pattern

- Requirement: Given $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_P$
  - Design $\mathbf{W}$ such that
    - $sign(\mathbf{W}\mathbf{y}_p) = \mathbf{y}_p$ for all target patterns
    - There are no other *binary* vectors for which this holds

- What is the largest number of patterns that can be stored?

Storing K orthogonal patterns

- Simple solution: Design $\mathbf{W}$ such that $\mathbf{y}_1$, $\mathbf{y}_2, \dots, \mathbf{y}_K$ are the Eigen vectors of $\mathbf{W}$
  - Let $\mathbf{Y} = [\mathbf{y}_1\ \mathbf{y}_2 \dots \mathbf{y}_K]$

$$\mathbf{W} = \mathbf{Y}\Lambda\mathbf{Y}^T$$

  - $\lambda_1, \dots, \lambda_K$ are positive
  - For $\lambda_1 = \lambda_2 = \lambda_K = 1$ this is exactly the Hebbian rule
- The patterns are provably stationary

Hebbian rule

- In reality
  - Let $\mathbf{Y} = [\mathbf{y}_1\ \mathbf{y}_2 \ldots \mathbf{y}_K\ \mathbf{r}_{K+1}\ \mathbf{r}_{K+2} \ldots \mathbf{r}_N]$

$$\mathbf{W} = \mathbf{Y}\Lambda\mathbf{Y}^T$$

  - $\mathbf{r}_{K+1}\ \mathbf{r}_{K+2} \ldots \mathbf{r}_N$ are orthogonal to $\mathbf{y}_1\ \mathbf{y}_2 \ldots \mathbf{y}_K$
  - $\lambda_1 = \lambda_2 = \lambda_K = 1$
  - $\lambda_{K+1}, \ldots, \lambda_N = 0$

Storing N orthogonal patterns

- When we have N orthogonal (or near orthogonal) patterns $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_N$
  - $Y = [\mathbf{y}_1\ \mathbf{y}_2 \ldots \mathbf{y}_N]$

$$\mathbf{W} = \mathbf{Y}\Lambda\mathbf{Y}^T$$

  - $\lambda_1 = \lambda_2 = \lambda_N = 1$
- The Eigen vectors of $\mathbf{W}$ span the space
- Also, for any $\mathbf{y}_k$

$$\mathbf{W}\mathbf{y}_k = \mathbf{y}_k$$

Storing N orthogonal patterns

- The N orthogonal patterns $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_N$ *span the space*
- Any pattern $\mathbf{y}$ can be written as

$$\mathbf{y} = a_1\mathbf{y}_1 + a_2\mathbf{y}_2 + \cdots + a_N\mathbf{y}_N$$

$$\mathbf{Wy} = a_1\mathbf{Wy}_1 + a_2\mathbf{Wy}_2 + \cdots + a_N\mathbf{Wy}_N$$

$$= a_1\mathbf{y}_1 + a_2\mathbf{y}_2 + \cdots + a_N\mathbf{y}_N = \mathbf{y}$$

- *All patterns are stationary*
  - Everything is a stationary memory
  - ***Completely useless network***

Storing *K* orthogonal patterns

- Even if we store fewer than N patterns
  - Let $Y = [\mathbf{y}_1\ \mathbf{y}_2 \ldots \mathbf{y}_K\ \mathbf{r}_{K+1}\ \mathbf{r}_{K+2} \ldots \mathbf{r}_N]$

$$W = Y\Lambda Y^T$$

  - $\mathbf{r}_{K+1}\ \mathbf{r}_{K+2} \ldots \mathbf{r}_N$ are orthogonal to $\mathbf{y}_1\ \mathbf{y}_2 \ldots \mathbf{y}_K$
  - $\lambda_1 = \lambda_2 = \lambda_K = 1$
  - $\lambda_{K+1}, \ldots, \lambda_N = 0$
- Any pattern that is *entirely* in the subspace spanned by $\mathbf{y}_1\ \mathbf{y}_2 \ldots \mathbf{y}_K$ is also stable (same logic as earlier)
- Only patterns that are *partially* in the subspace spanned by $\mathbf{y}_1\ \mathbf{y}_2 \ldots \mathbf{y}_K$ are unstable
  - Get projected onto subspace spanned by $\mathbf{y}_1\ \mathbf{y}_2 \ldots \mathbf{y}_K$

Problem with Hebbian Rule

- Even if we store fewer than N patterns
  - Let $Y = [\mathbf{y}_1\ \mathbf{y}_2 \ldots \mathbf{y}_K\ \mathbf{r}_{K+1}\ \mathbf{r}_{K+2} \ldots \mathbf{r}_N]$

$$W = Y \Lambda Y^T$$

  - $\mathbf{r}_{K+1}\ \mathbf{r}_{K+2} \ldots \mathbf{r}_N$ are orthogonal to $\mathbf{y}_1\ \mathbf{y}_2 \ldots \mathbf{y}_K$
  - $\lambda_1 = \lambda_2 = \lambda_K = 1$
- Problems arise because Eigen values are all 1.0
  - Ensures stationarity of vectors in the subspace
  - All stored patterns are equally important
  - What if we get rid of this requirement?

Hebbian rule and general (non-orthogonal) vectors

$$w_{ji} = \sum_{p \in \{p\}} y_i^p y_j^p$$

- What happens when the patterns are *not* orthogonal
- What happens when the patterns are presented *more* than once
  – Different patterns presented different numbers of times
  – Equivalent to having unequal Eigen values..
- Can we predict the evolution of any vector $\mathbf{y}$
  – Hint: For real valued vectors, use Lanczos iterations
    - Can write $\mathbf{Y}_P = \mathbf{U}_P \Lambda \mathbf{V}_p^T, \rightarrow \mathbf{W} = \mathbf{U}_P \Lambda^2 \mathbf{U}_p^T$
  – Tougher for binary vectors (NP)

The bottom line

- With a network of N units (i.e. N-bit patterns)
- The maximum number of stationary patterns is actually *exponential* in N
  - McElice and Posner, 84'
  - E.g. when we had the Hebbian net with N orthogonal base patterns, *all* patterns are stationary

- For a *specific* set of K patterns, we can *always* build a network for which all K patterns are stable provided $K \leq N$
  - Mostafa and St. Jacques 85'
    - For large N, the upper bound on K is actually N/4logN
      - McElice et. Al. 87'
  - **But this may come with many "parasitic" memories**

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Can we do something about this?

Story so far

- Hopfield nets with N neurons can store up to 0.14N random patterns through Hebbian learning with 0.996 probability of recall
  - The recalled patterns are the Eigen vectors of the weights matrix with the highest Eigen values

- Hebbian learning assumes all patterns to be stored are equally important
  - For orthogonal patterns, the patterns are the Eigen vectors of the constructed weights matrix
  - All Eigen values are identical

- In theory the number of stationary states in a Hopfield network can be exponential in N

- The number of *intentionally* stored patterns (stationary *and* stable) can be as large as N
  - But comes with many parasitic memories