

Deep Neural Networks

Convolutional Networks III

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Fall 2025

Attendance: @757

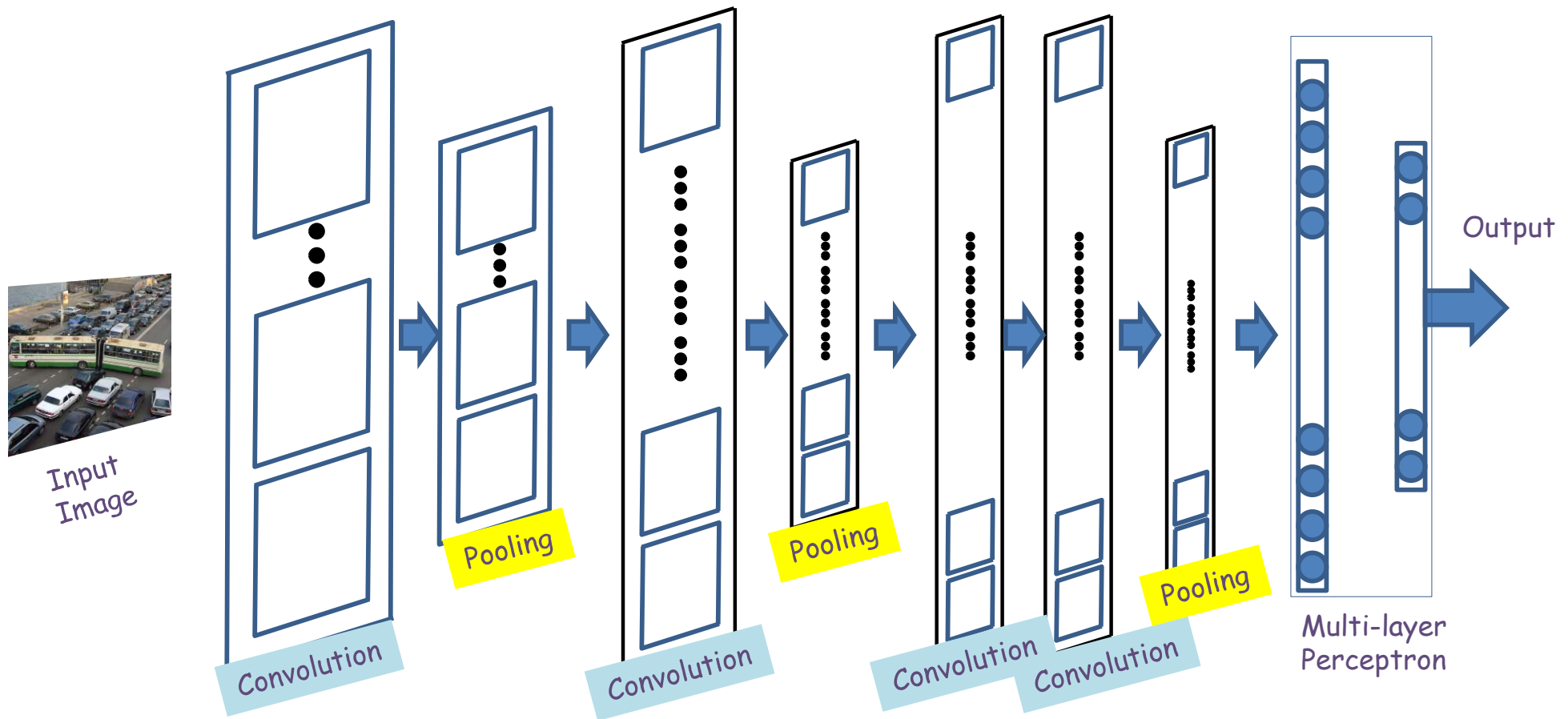
Outline

- Quick recap
- Back propagation through a CNN
- Modifications: Transposition, scaling, rotation and deformation invariance
- Segmentation and localization
- Some success stories
- Some advanced architectures
 - Resnet
 - Densenet

Story so far

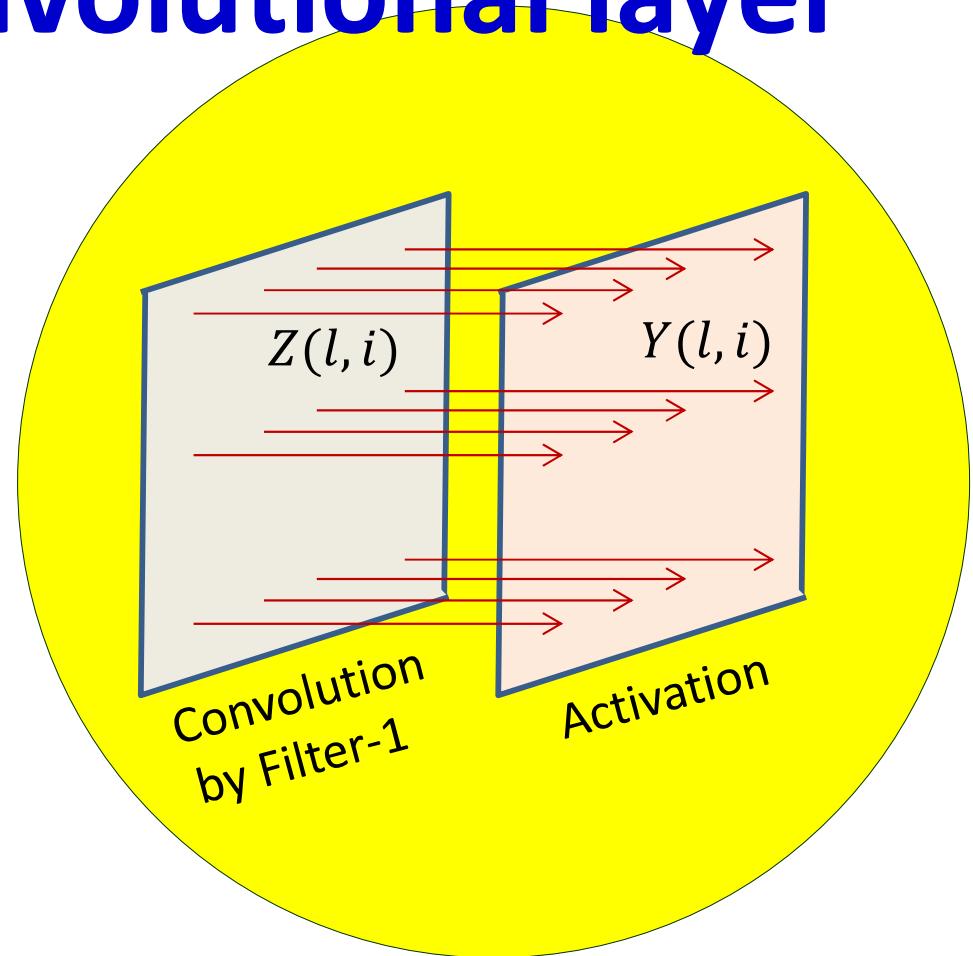
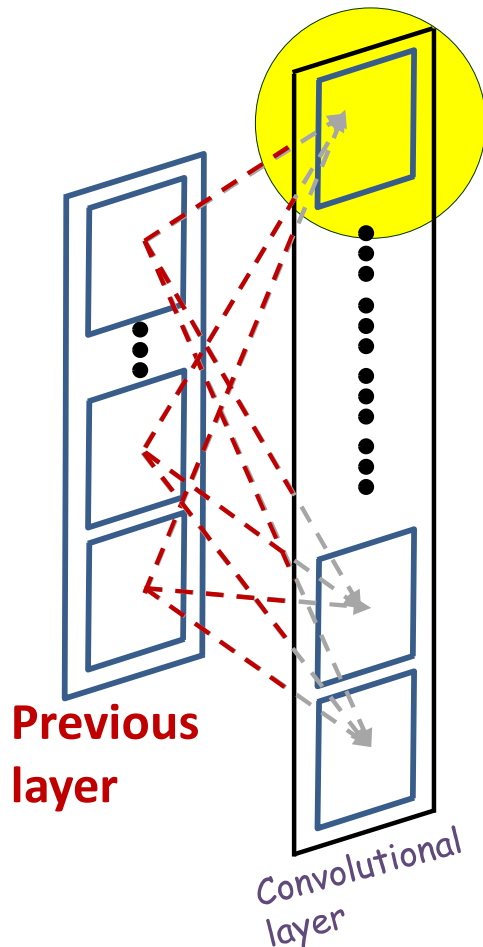
- Pattern classification tasks such as “does this picture contain a cat”, or “does this recording include HELLO” are best performed by scanning for the target pattern
- Scanning an input with a network and combining the outcomes is equivalent to scanning with individual neurons hierarchically
 - First level neurons scan the input
 - Higher-level neurons scan the “maps” formed by lower-level neurons
 - A final “decision” unit or layer makes the final decision
 - Deformations in the input can be handled by “pooling”
- For 2-D (or higher-dimensional) scans, the structure is called a Convolutional Neural Network
- For 1-D scan along time, it is called a Time-delay neural network

Recap: The general architecture of a convolutional neural network



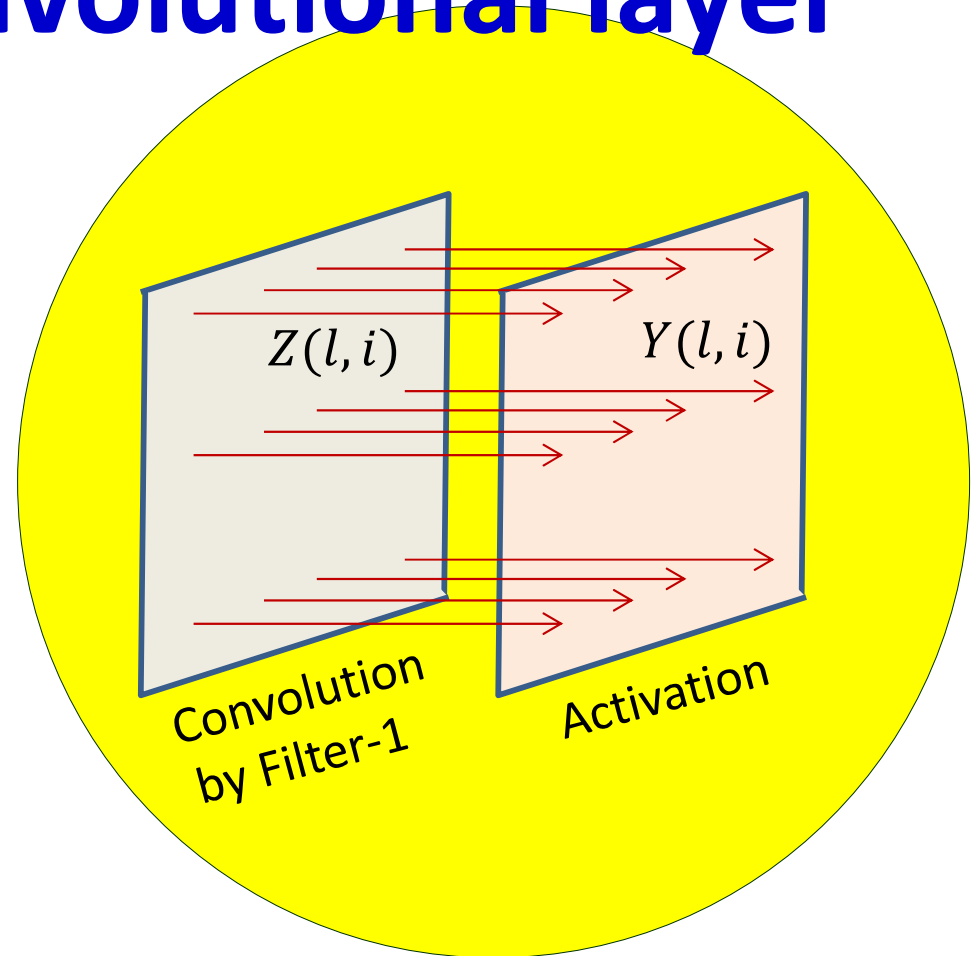
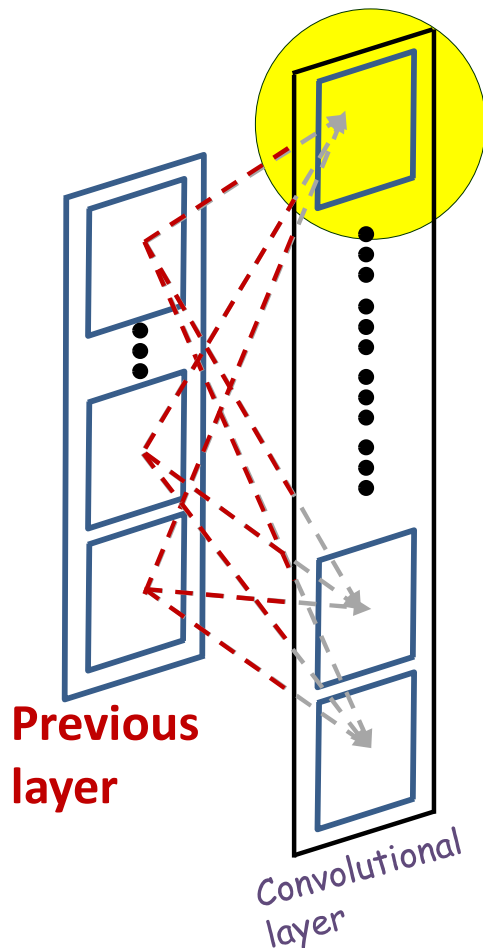
- A convolutional neural network comprises of “convolutional” and optional “pooling” layers
- Followed by an MLP with one or more layers

Recap: A convolutional layer



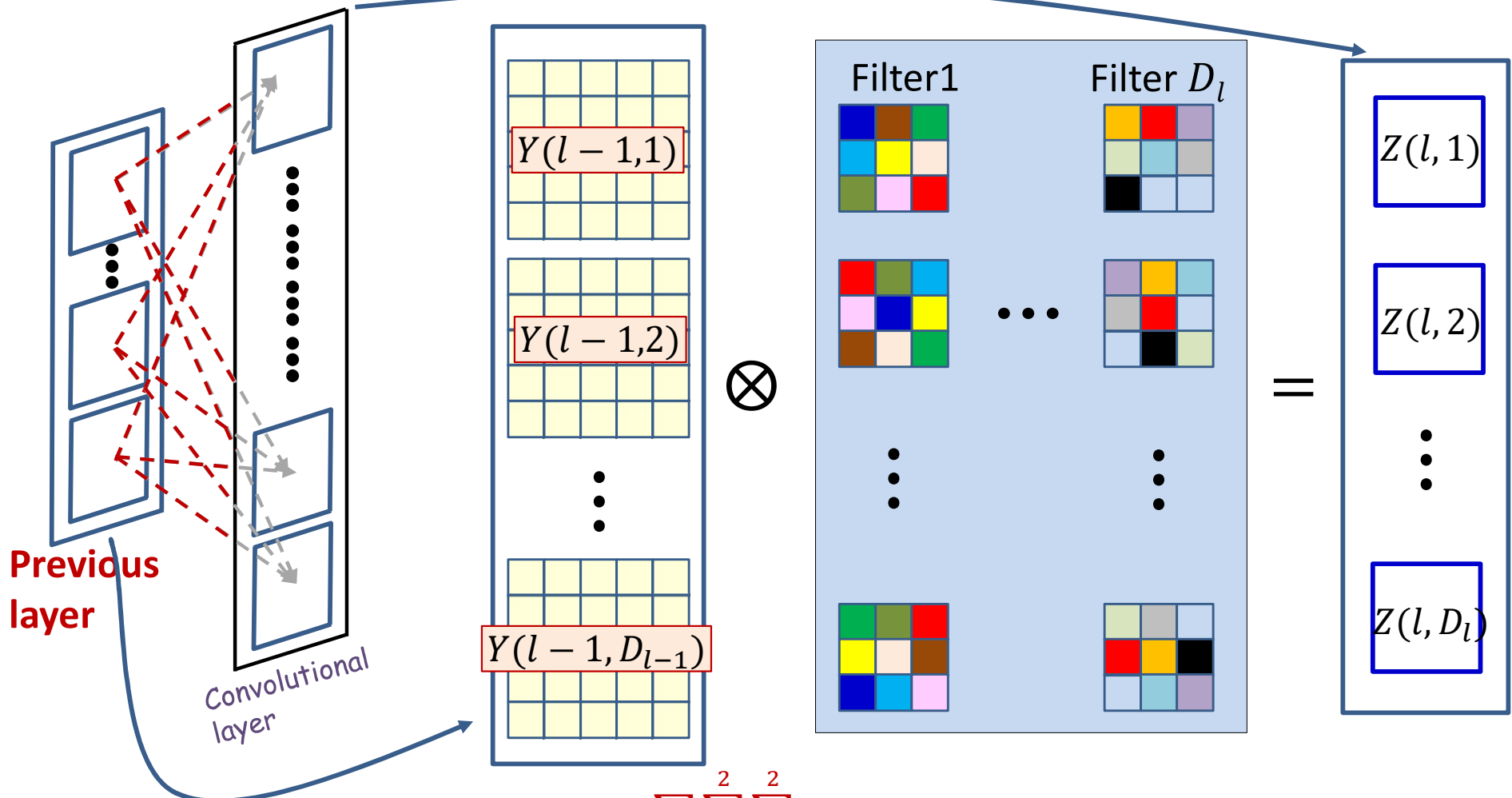
- The computation of each output map has two stages
 - Computing an *affine* map, by *convolving* a *filter* (representing a pattern of weights) over maps in the previous layer
 - Each affine map has, associated with it, a **learnable filter**
 - An *activation* that operates *point-wise* on the output of the convolution

Recap: A convolutional layer



- The computation of each output map has two stages
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Recap: Convolution

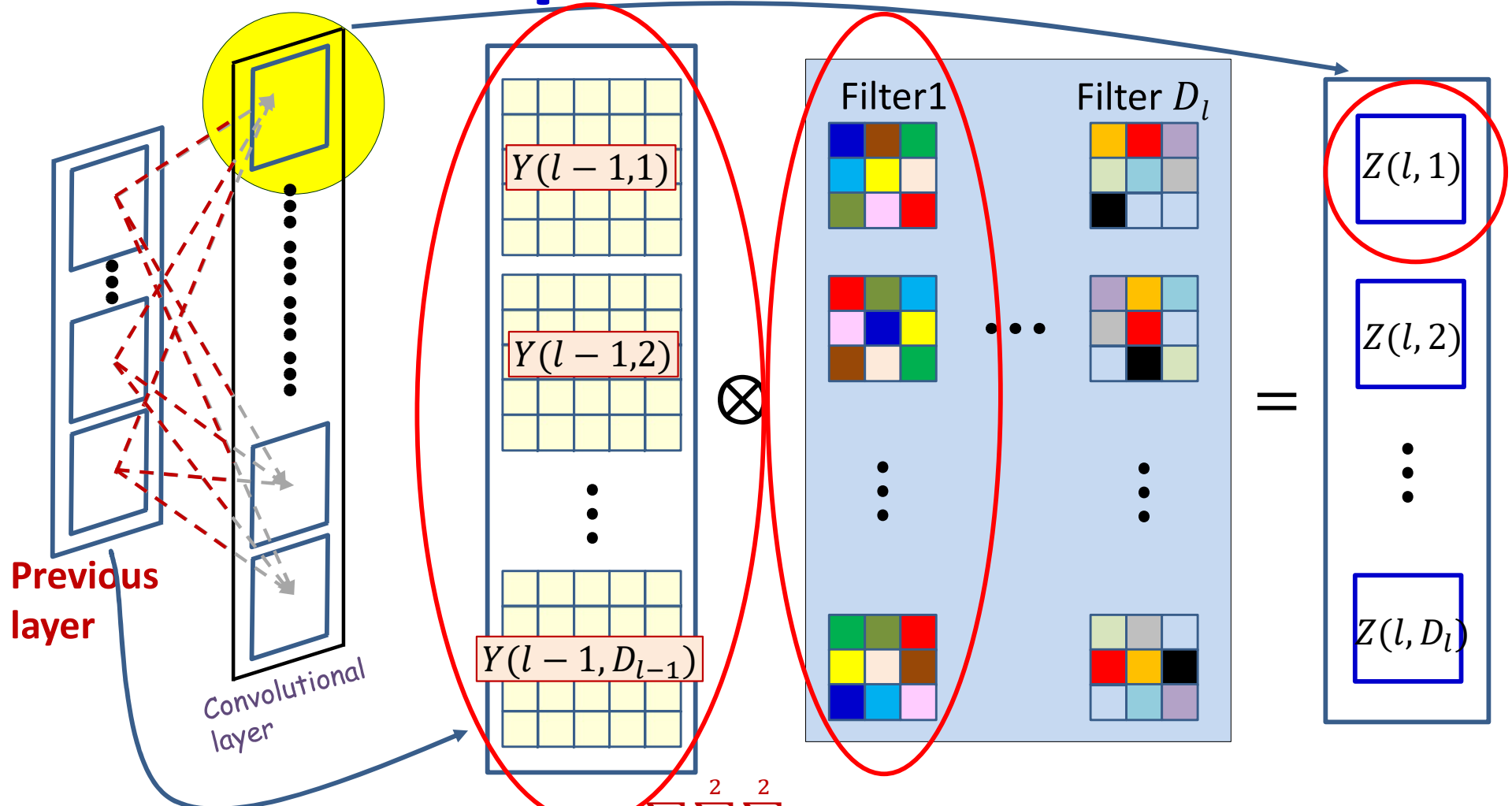


Caveat : 0-based indexing

$$z(l, n, x, y) = \sum_m \sum_{i=0}^2 \sum_{j=0}^2 w_l(m, n, i, j) y(l-1, m, x+i, y+j) + b_l(n)$$

- Each affine output map is computed from multiple input maps simultaneously
- There are as many weights (for each output map) as *size of the filter* x *no. of maps in previous layer*

Recap: Convolution

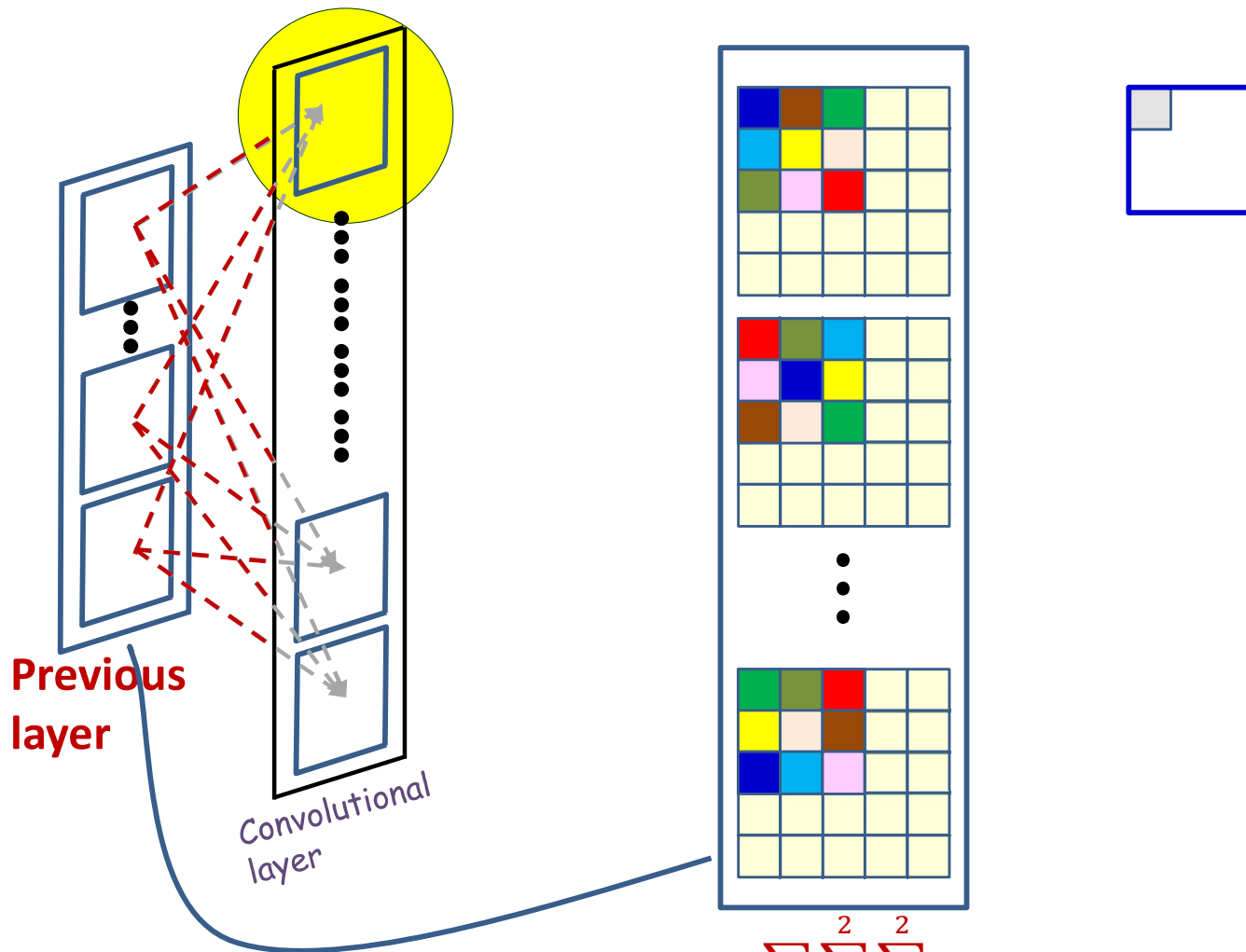


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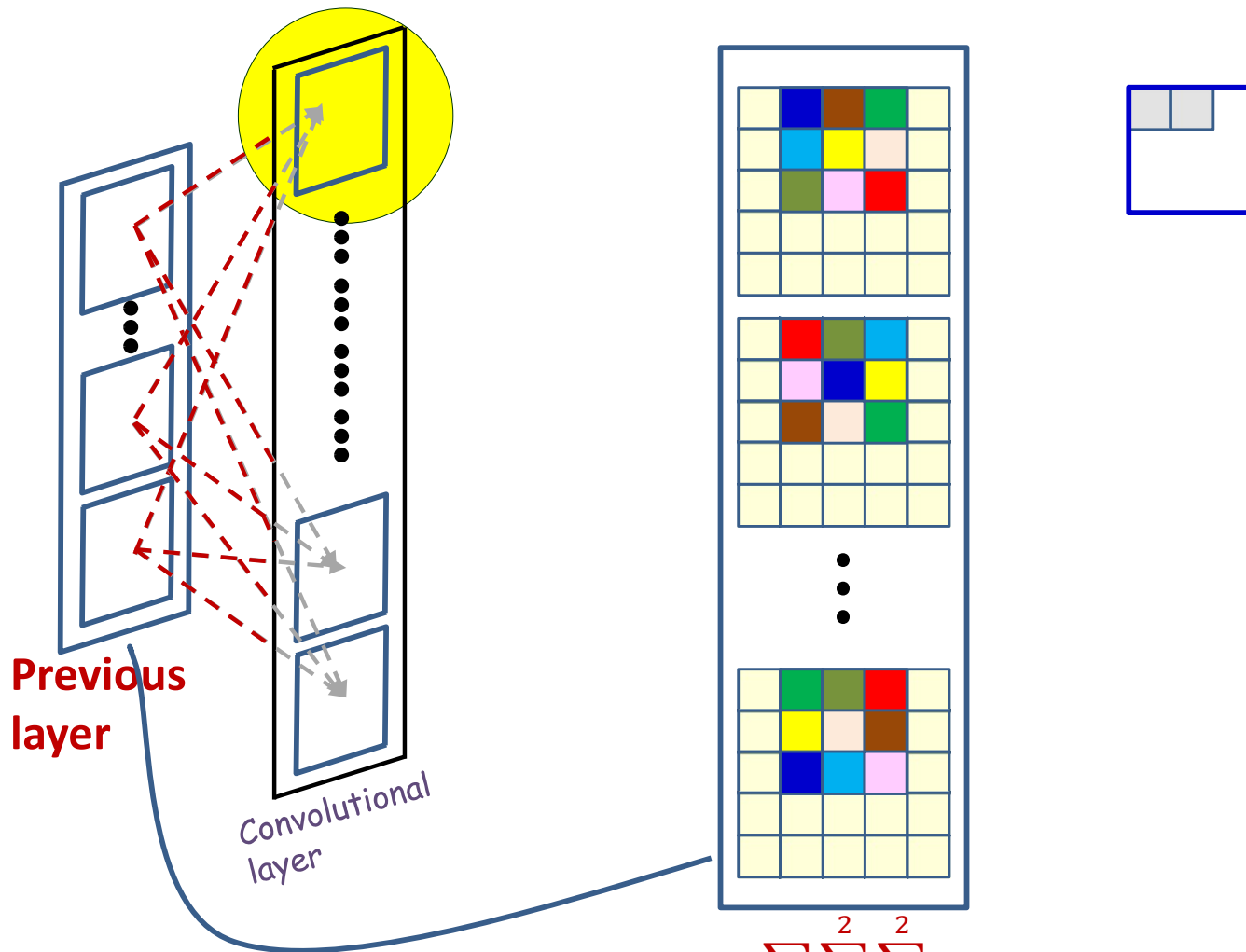
Recap: Convolution



$$z(l, n, x, y) = \sum_m \sum_{i=0}^2 \sum_{j=0}^2 w_l(m, n, i, j) y(l-1, m, x+i, y+j) + b_l(n)$$

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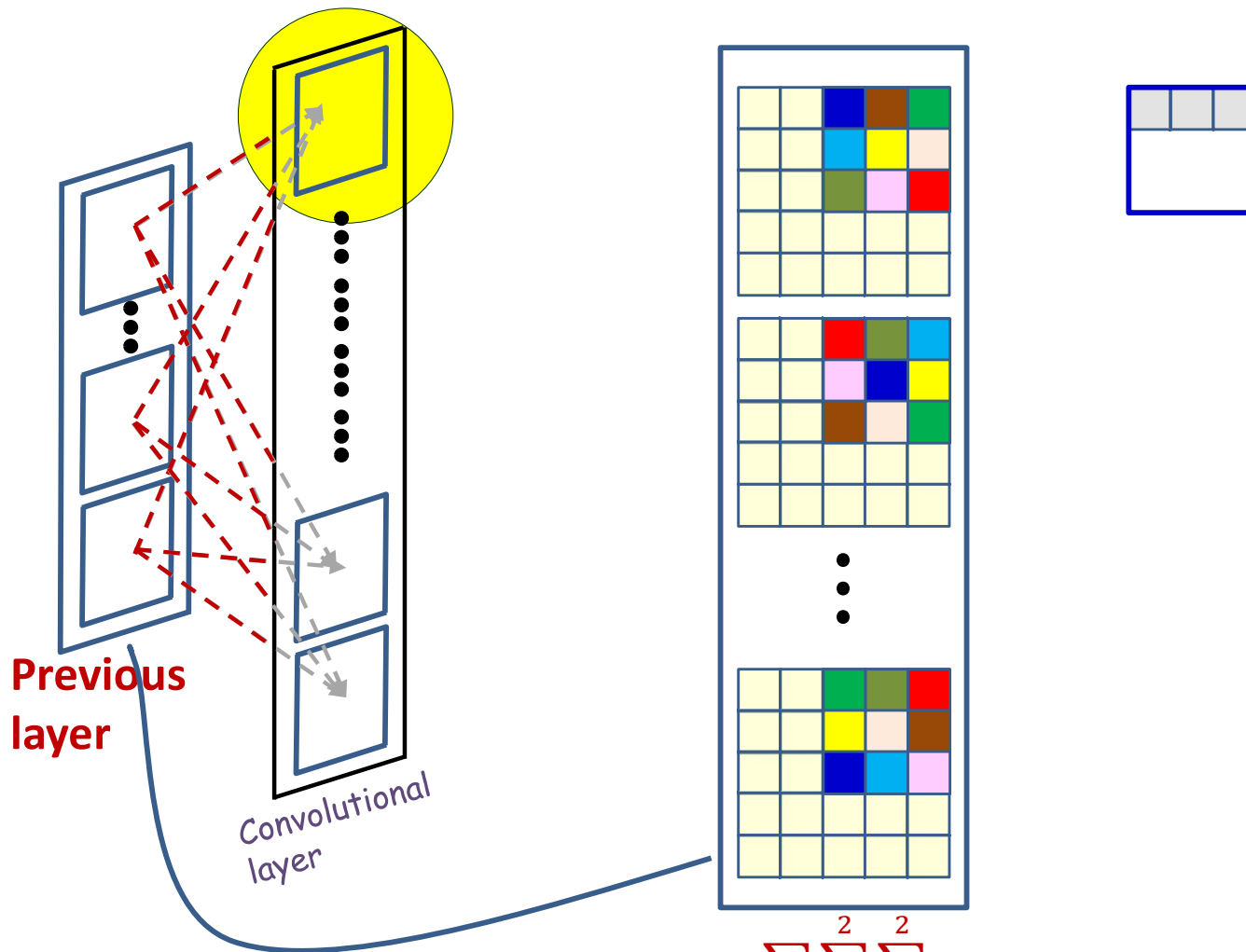
Recap: Convolution



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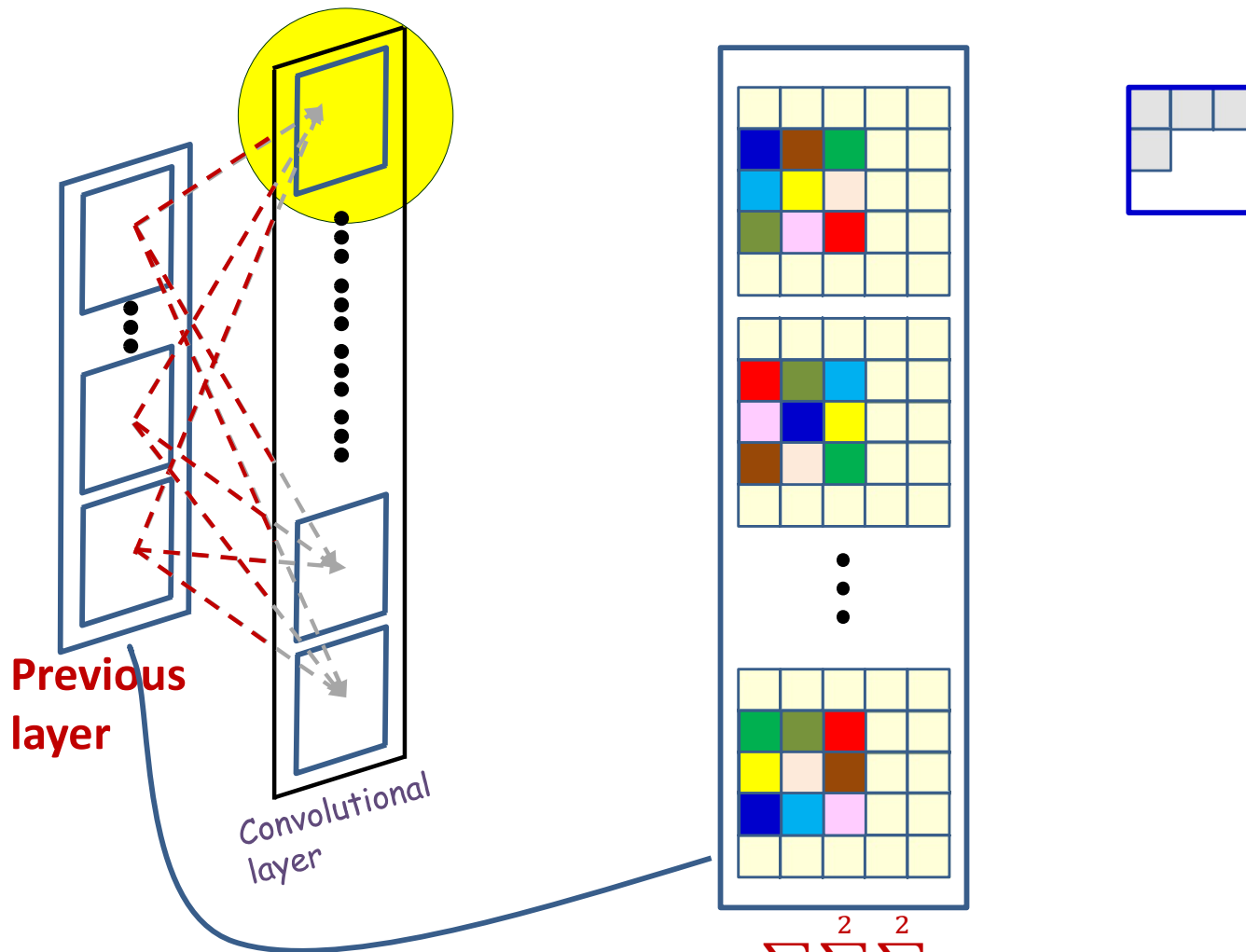
Recap: Convolution



$$z(l, n, x, y) = \sum_m \sum_{i=0}^2 \sum_{j=0}^2 w_l(m, n, i, j) y(l-1, m, x+i, y+j) + b_l(n)$$

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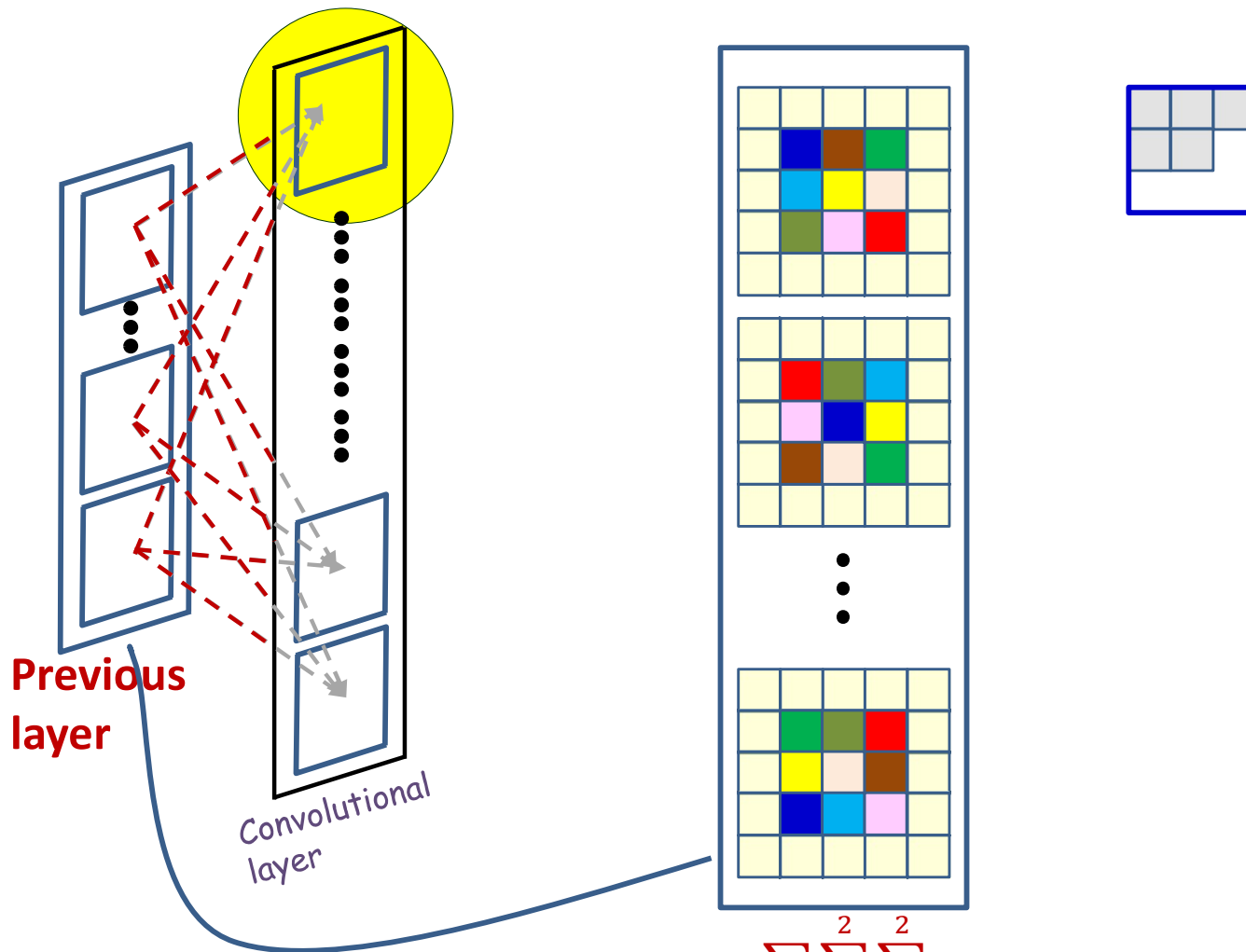
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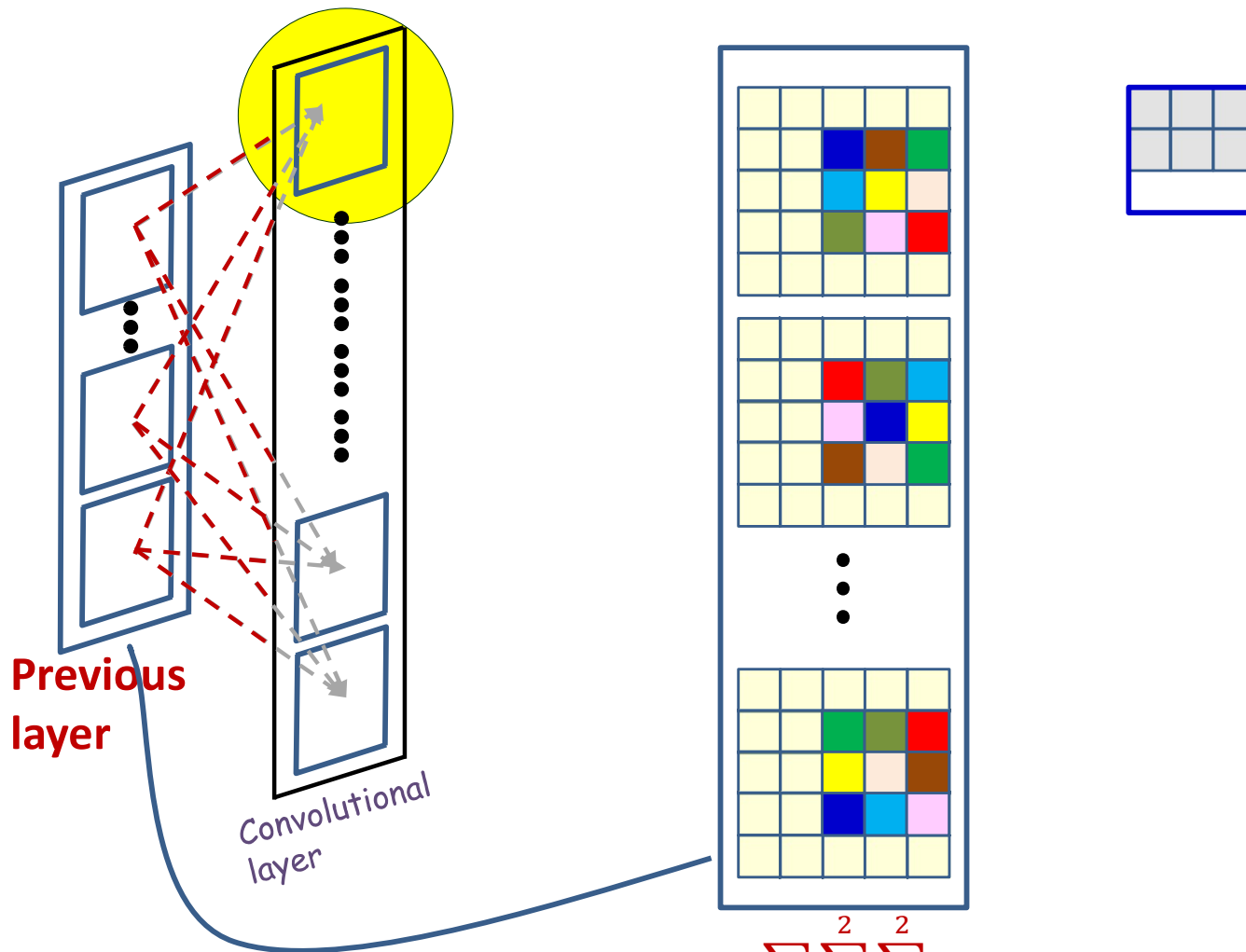
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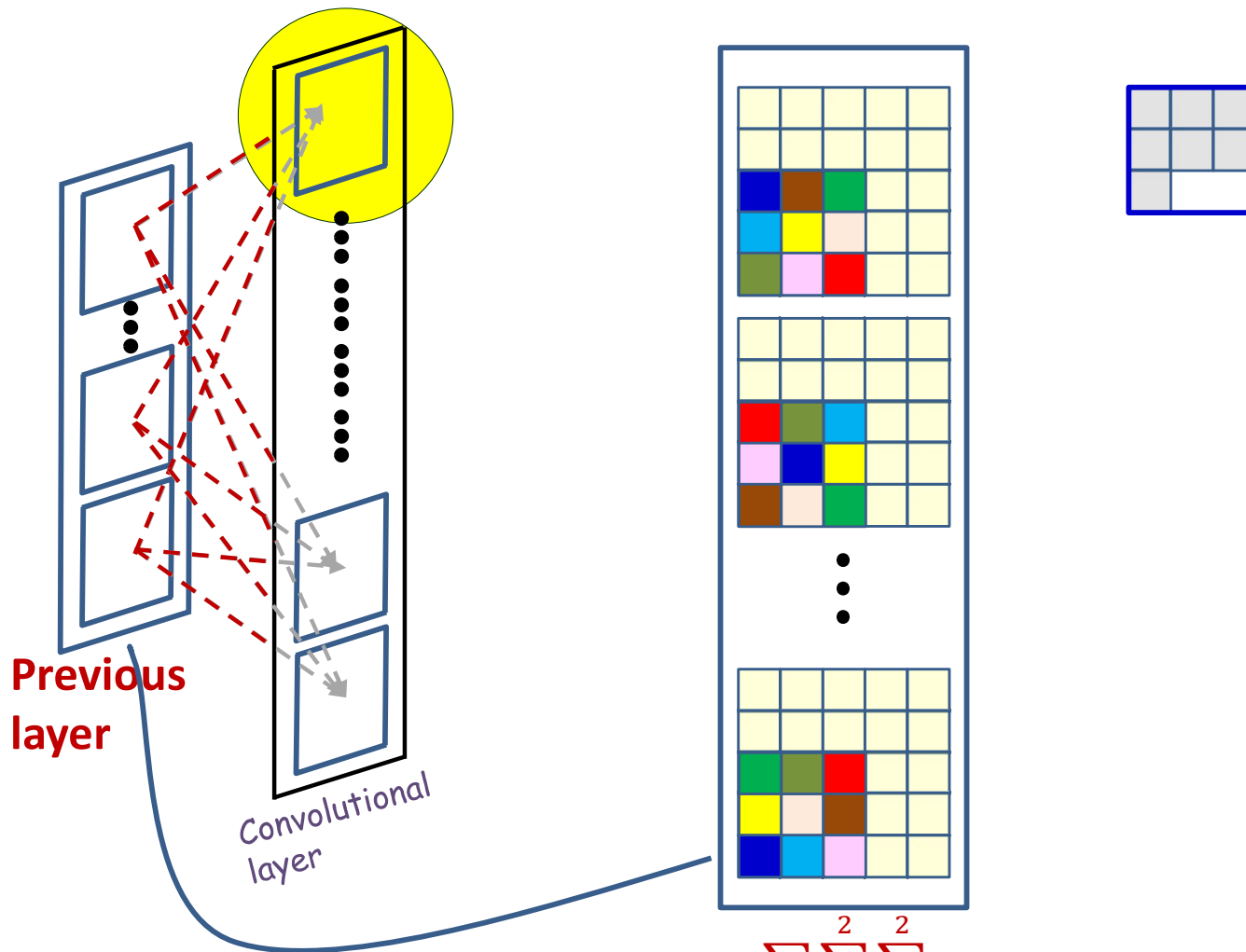
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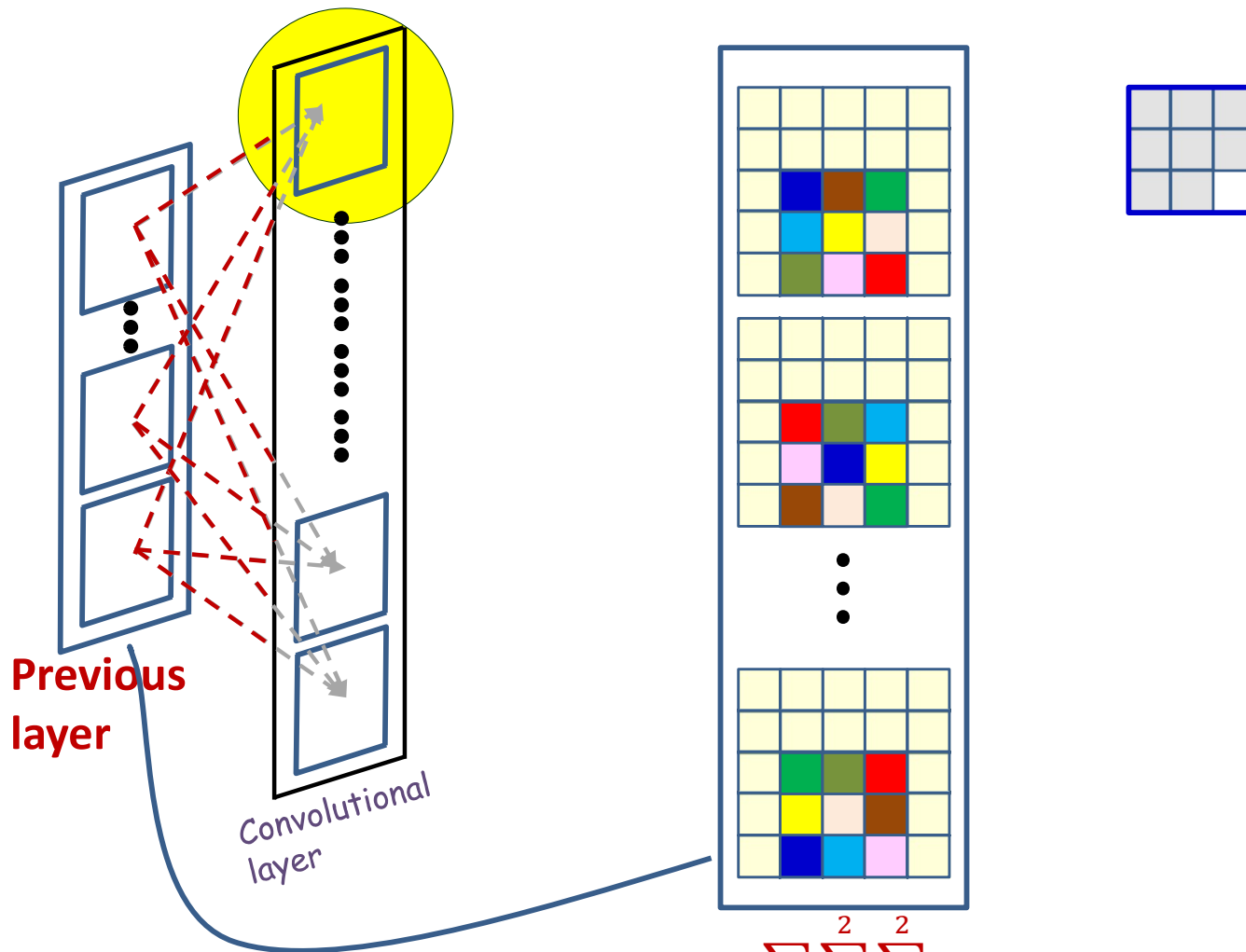
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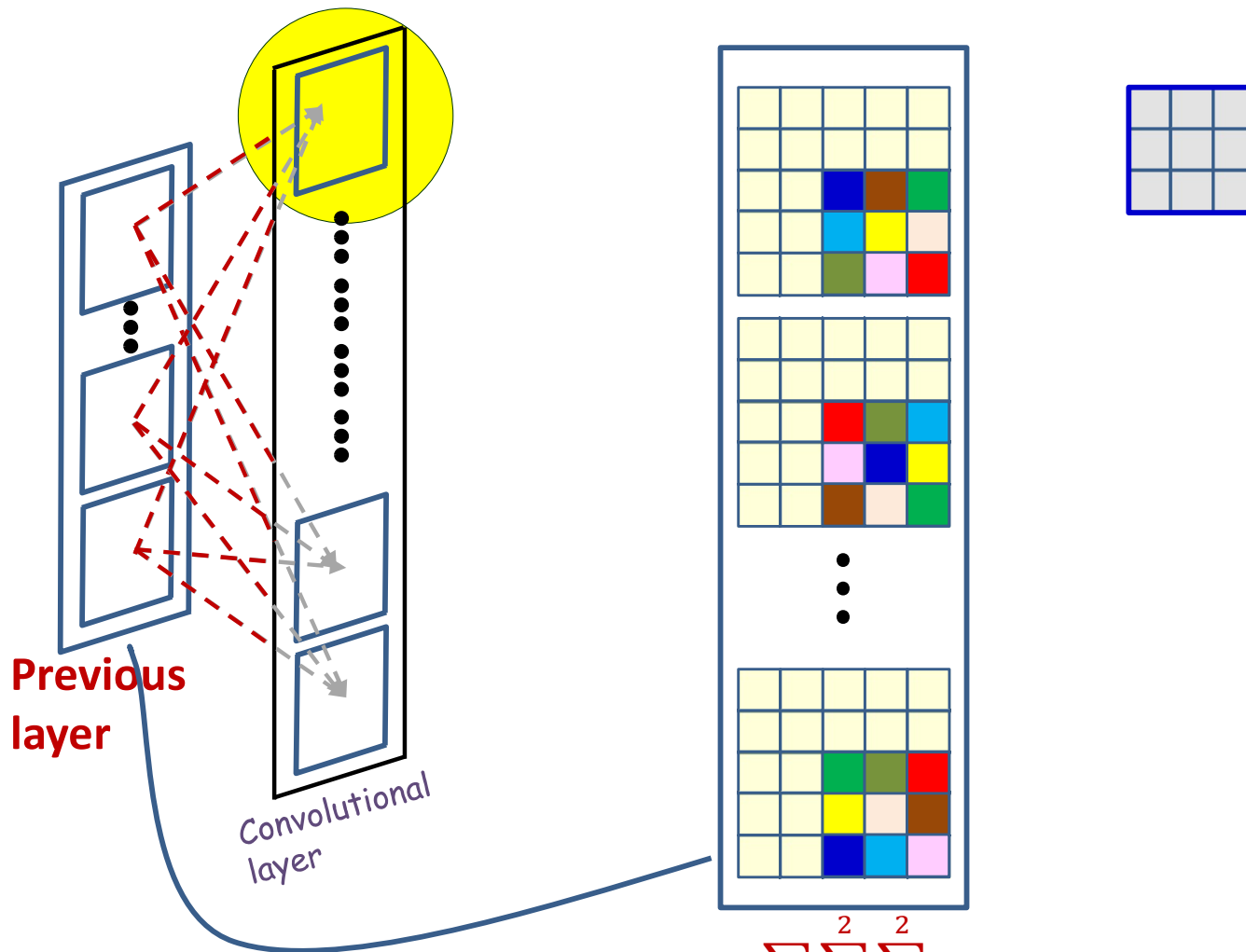
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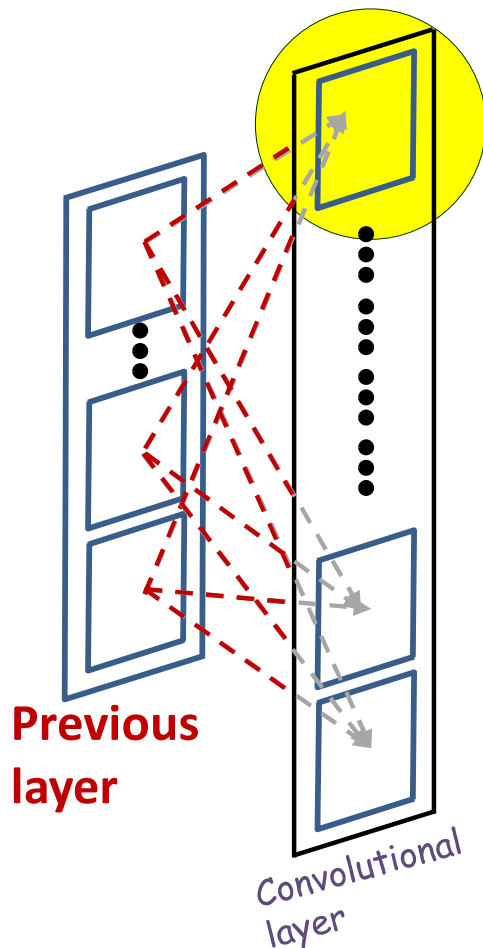
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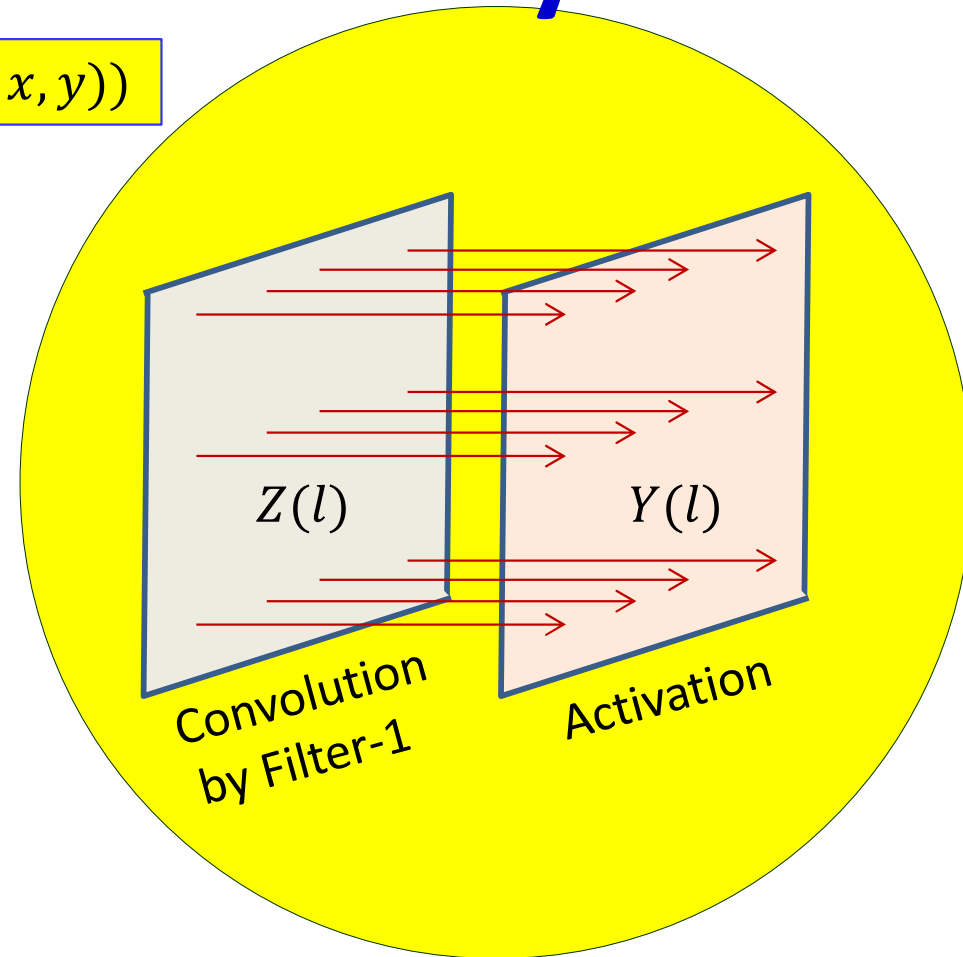
$$z(l, n, x, y) = \sum_m \sum_{i=0}^2 \sum_{j=0}^2 w_l(m, n, i, j) y(l-1, m, x+i, y+j) + b_l(n)$$

- Each affine output is computed from multiple input maps simultaneously
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Recap: A convolutional layer



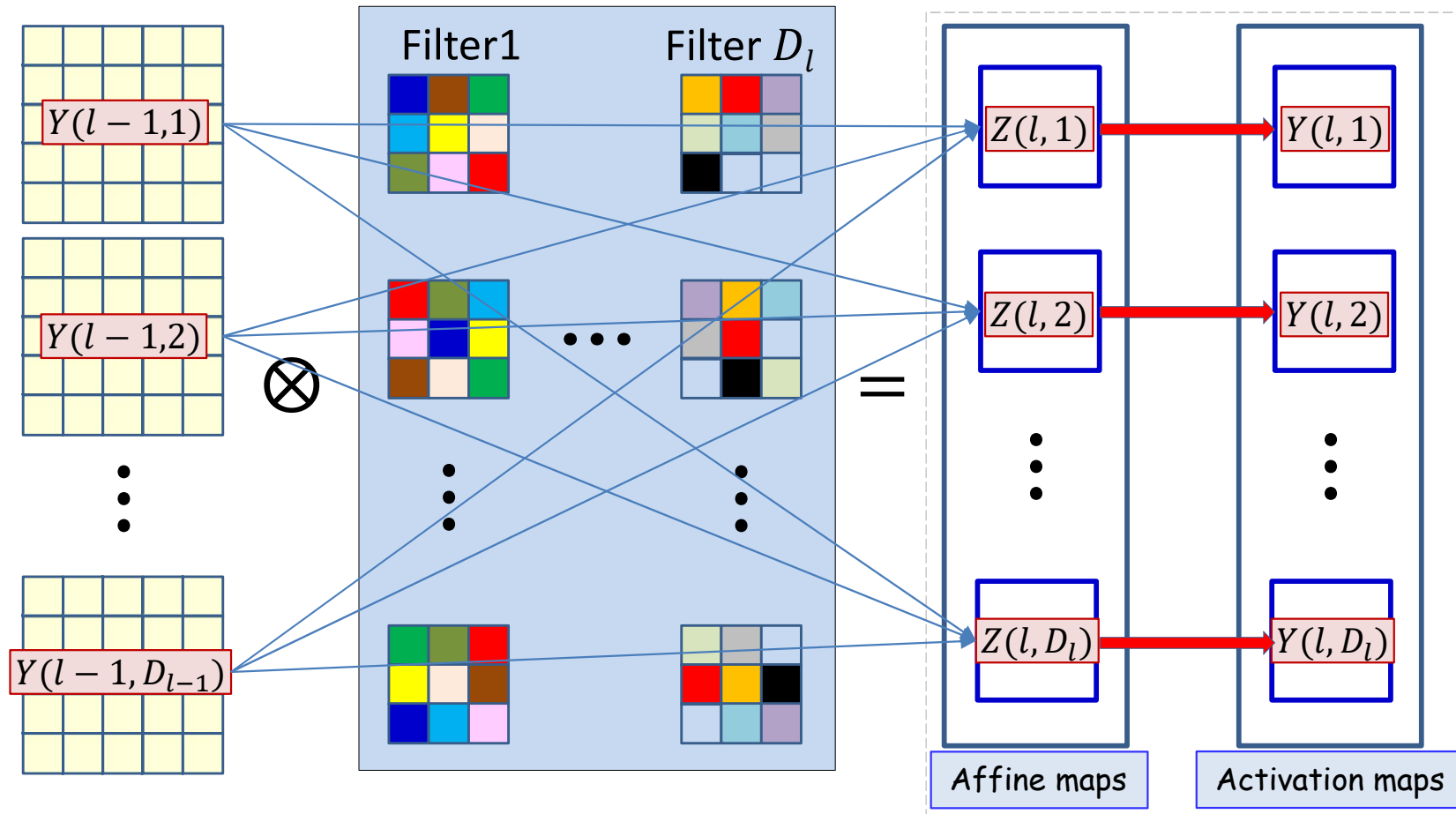
$$y(l, i, x, y) = f(z(l, i, x, y))$$



- The computation of each output map has two stages
 - Computing an *affine* map, by *convolution* of a *filter* (representing a pattern of weights) over maps in the previous layer
 - Each affine map has, associated with it, a **learnable filter**
 - An *activation* that operates on the output of the convolution

Convolution layer: A more explicit illustration

$$y(l, i, x, y) = f(z(l, i, x, y))$$



- Input maps $Y(l-1,*)$ are convolved with several filters to generate the affine maps $Z(l,*)$
 - Each filter consists of a set of square patterns of weights, with one set for each map in $Y(l-1,*)$
 - We get one affine map per filter
- A *point-wise* activation function $f(z)$ is applied to each map in $Z(l,*)$ to produce the activation maps $Y(l,*)$

Pseudocode: Vector notation

The weight $W(l, j)$ is a 3D $D_{l-1} \times K_l \times K_l$ tensor

$Y(0) = \text{Image}$

for $l = 1:L$ **# layers operate on vector at (x,y)**

 for $x = 1:W_{l-1}-K_l+1$

 for $y = 1:H_{l-1}-K_l+1$

 for $j = 1:D_l$

segment = $Y(l-1, :, x:x+K_l-1, y:y+K_l-1)$ **#3D tensor**

$z(l, j, x, y) = W(l, j) \cdot \text{segment} + b(l, j)$ **#tensor prod.**

$Y(l, j, x, y) = \text{activation}(z(l, j, x, y))$

$Y = \text{softmax}(\{Y(L, :, :, :)\})$

Pseudocode has 1-based indexing

Poll 1: @756

Select all true statements about a convolution layer.

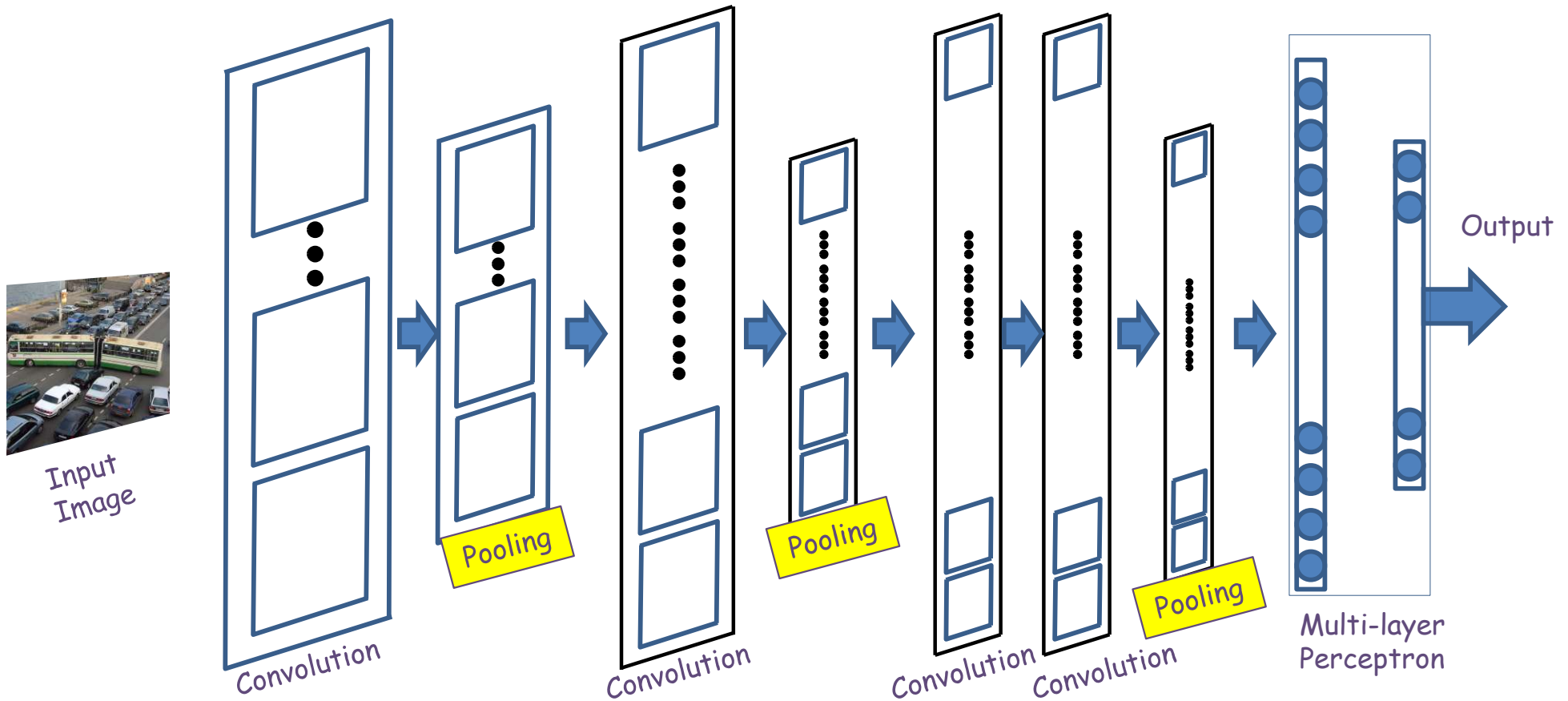
- The number of “channels” in any filter equals the number of input maps (output maps from the previous layer)
- The number of “channels” in any filter equals the number of output maps (affine maps output by the layer)
- The number of filters equals the number of input maps
- The number of filters equals the number of output maps

Poll 1

Select all true statements about a convolution layer.

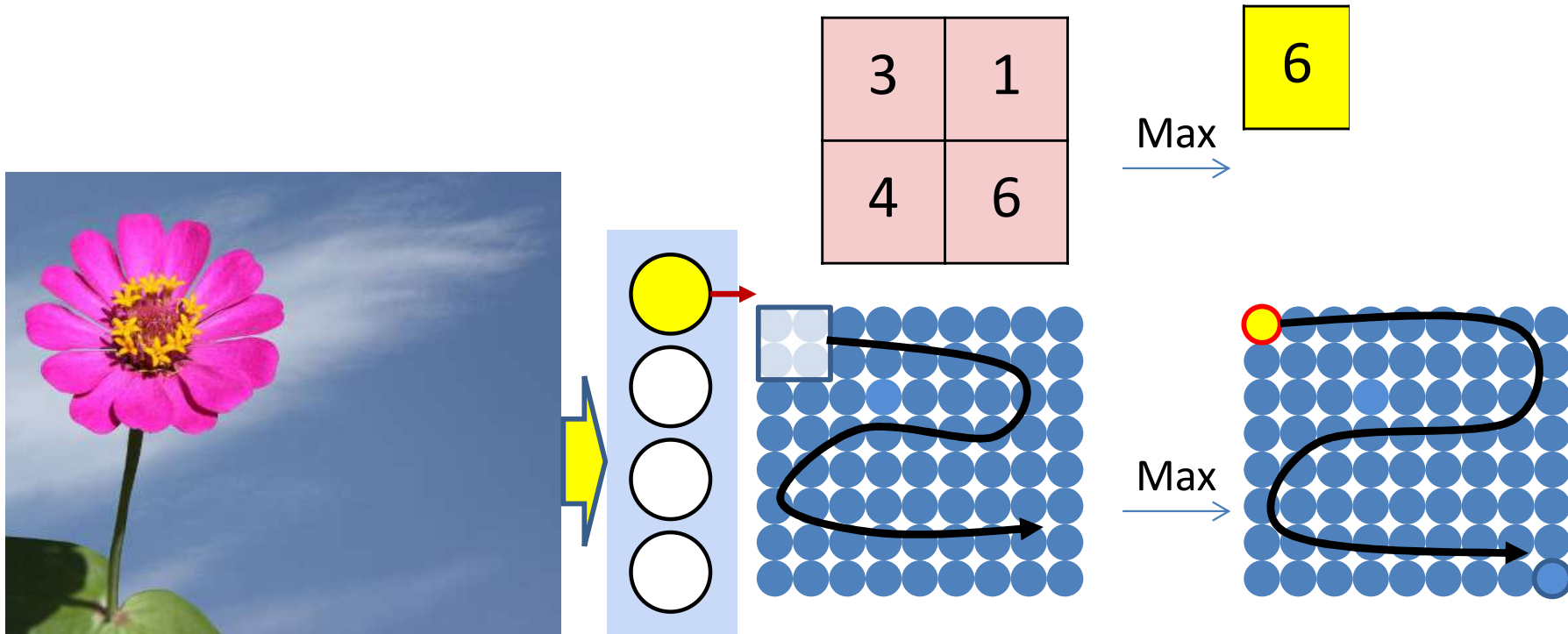
- **The number of “channels” in any filter equals the number of input maps (output maps from the previous layer)**
- The number of “channels” in any filter equals the number of output maps (affine maps output by the layer)
- The number of filters equals the number of input maps
- **The number of filters equals the number of output maps**

Pooling



- Convolutional (and activation) layers are followed intermittently by “pooling” layers
 - Often, they alternate with convolution, though this is not necessary

Recall: Max pooling



- Max pooling selects the largest from a pool of elements
- Pooling is performed by “scanning” the input with a “max-pooling filter”

Recap: Max Pooling layer at layer l

- a) Performed separately for every map (j).
 - *) Not combining multiple maps within a single max operation.
- b) Keeping track of location of max

Max pooling

```
for j = 1:D1
```

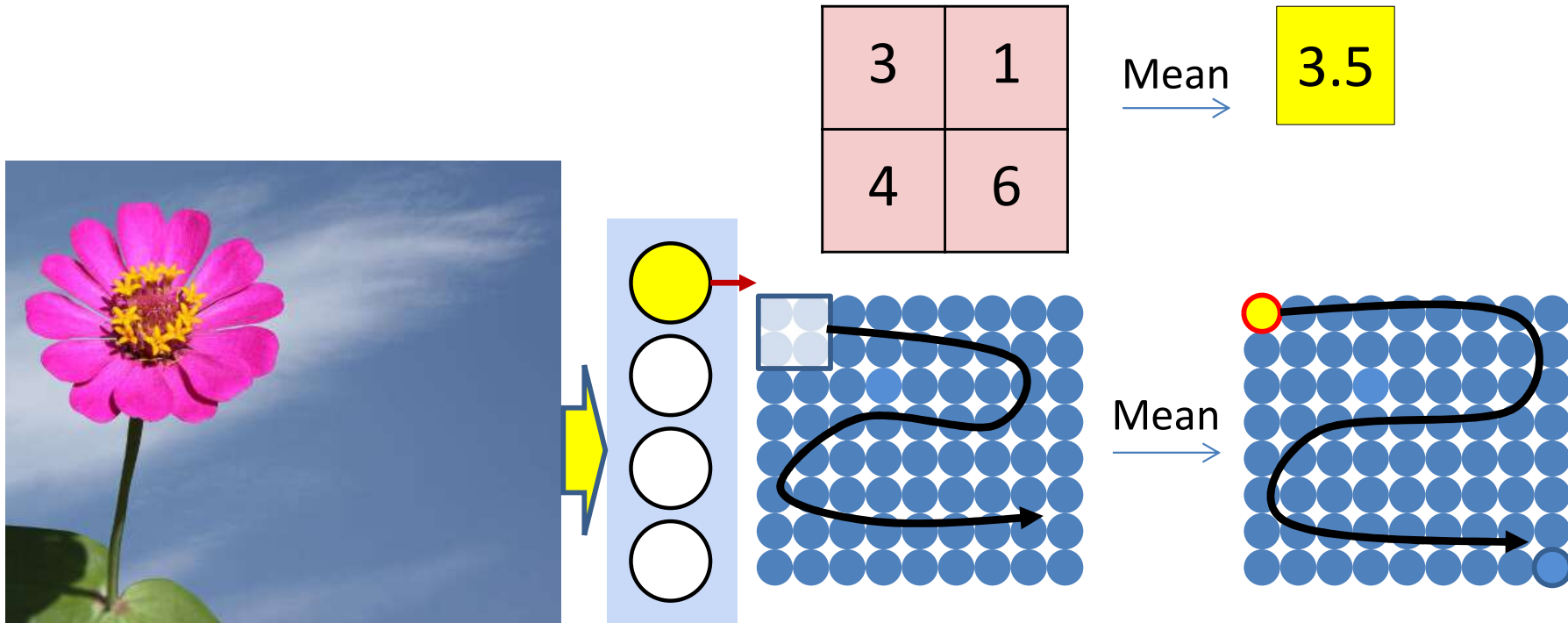
```
    for x = 1:Wl-1-K1+1
```

```
        for y = 1:Hl-1-K1+1
```

```
            pidx(l,j,x,y) = maxidx(Y(l-1,j,x:x+K1-1,y:y+K1-1))
```

```
            u(l,j,x,y) = Y(l-1,j,pidx(l,j,m,n))
```

Recall: Mean pooling



- Mean pooling computes the *mean* of the window of values
 - As opposed to the max of max pooling

Recap: Mean Pooling layer at layer l

a) Performed separately for every map (j)

Mean pooling

```
for j = 1:D1
```

```
    for x = 1:Wl-1-K1+1
```

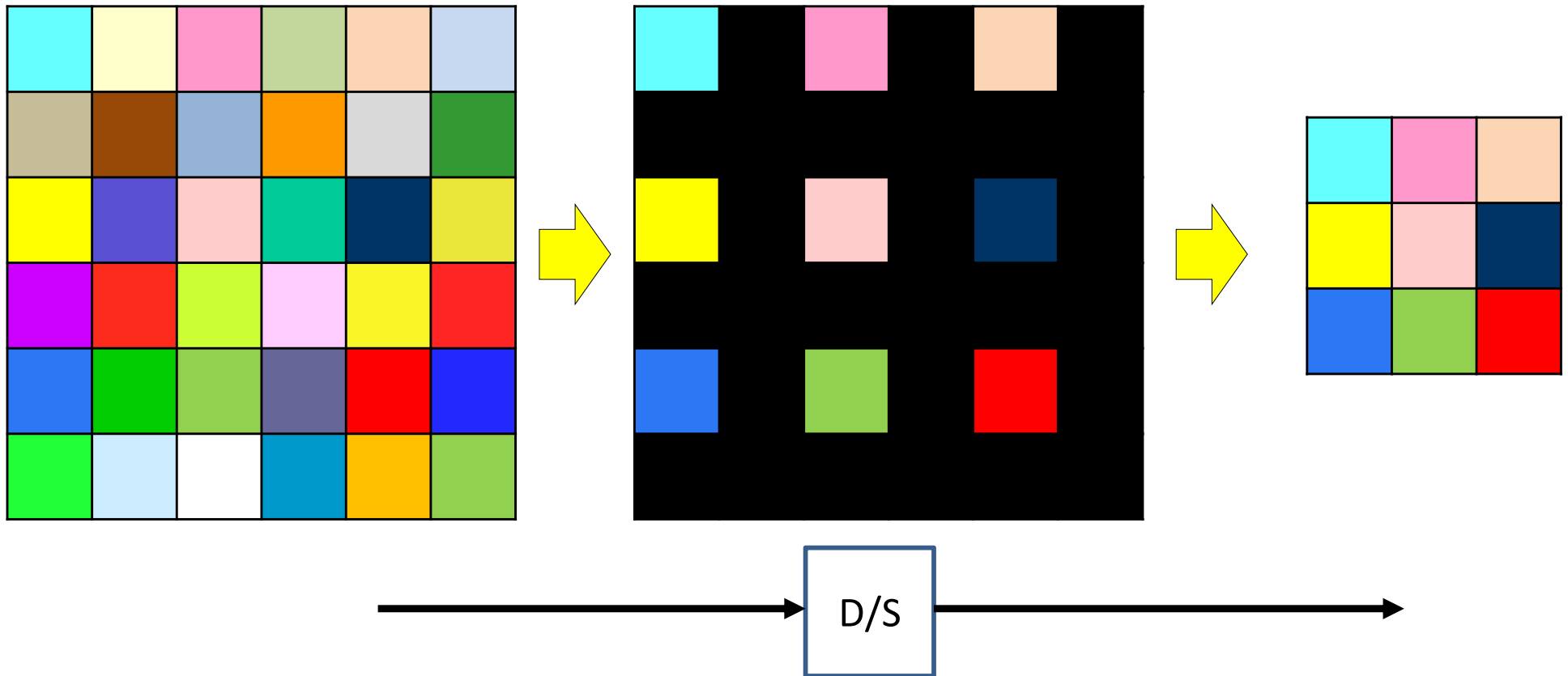
```
        for y = 1:Hl-1-K1+1
```

```
            u(l,j,x,y) = mean(Y(l-1,j,x:x+K1-1,y:y+K1-1))
```

Recap: Resampling

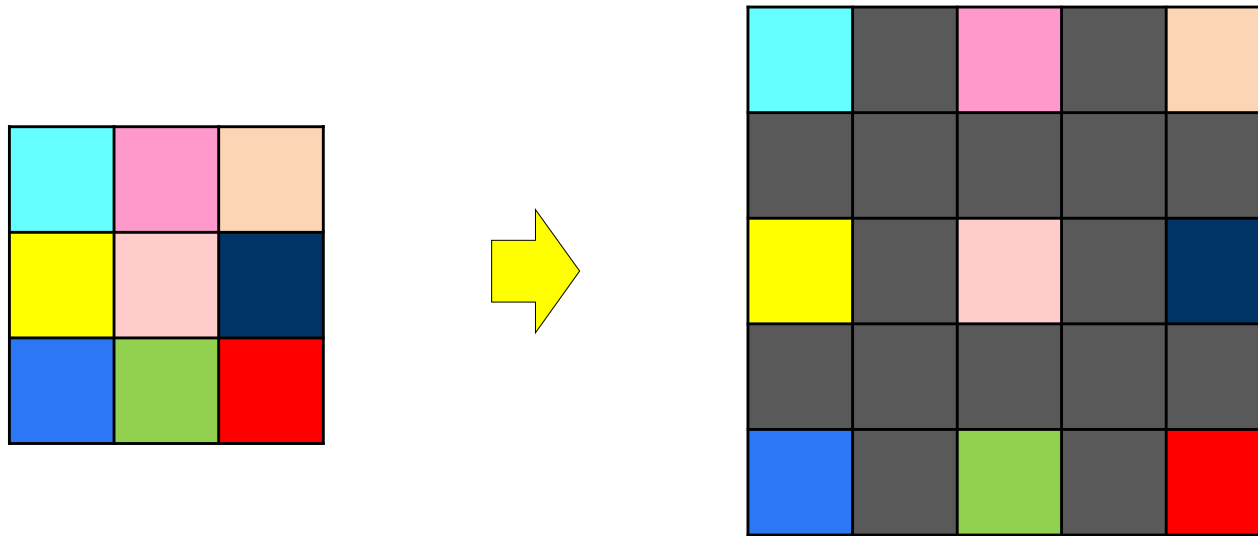
- We can also proportionately decrease or increase the size of the maps by dropping or inserting zeros
 - Downsampling: Drop $S-1$ rows/columns between rows/columns
 - Reduces the size of the maps by S on each side
 - Upsampling: Insert $S-1$ rows/columns of zeros between adjacent entries
 - Increases the size of the map by S on each side

The Downsampling Layer



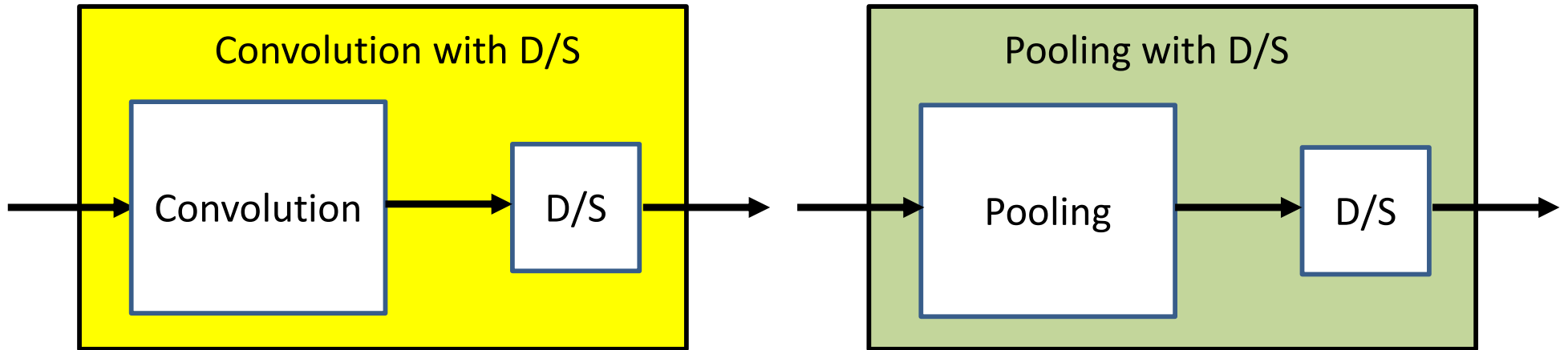
- A *downsampling* layer simply “drops” $S - 1$ of S rows and columns for every map in the layer
 - Effectively reducing the size of the map by factor S in every direction

The Upsampling Layer



- A *upsampling* (or dilation) layer simply introduces $S - 1$ rows and columns for every map in the layer
 - Effectively *increasing* the size of the map by factor S in every direction
- Used explicitly to increase the map size by a uniform factor

Downsampling in practice



- In practice, the downsampling is combined with the layers just before it by performing the operations with a stride > 1
 - Could be convolutional or pooling layers

Convolution with downsampling

The weight $W(l, j)$ is now a 4D $D_1 \times D_{1-1} \times K_1 \times K_1$ tensor

The product in blue is a tensor inner product with a scalar output

$Y(0) = \text{Image}$

for $l = 1:L$ # layers operate on vector at (x, y)

```
m = 1
```

```
for x = 1:S:W1-1-K1+1
```

```
    n = 1
```

```
    for y = 1:S:H1-1-K1+1
```

```
        segment = Y(l-1, :, x:x+K1-1, y:y+K1-1) #3D tensor
```

```
        z(l, :, m, n) = W(l).segment #tensor inner prod.
```

```
        Y(l, :, m, n) = activation(z(l, :, m, n))
```

```
        n++
```

```
    m++
```

STRIDE

Downsampled indices

```
Y = softmax( {Y(L, :, :, :)} )
```


Max Pooling with Downsampling

Max pooling

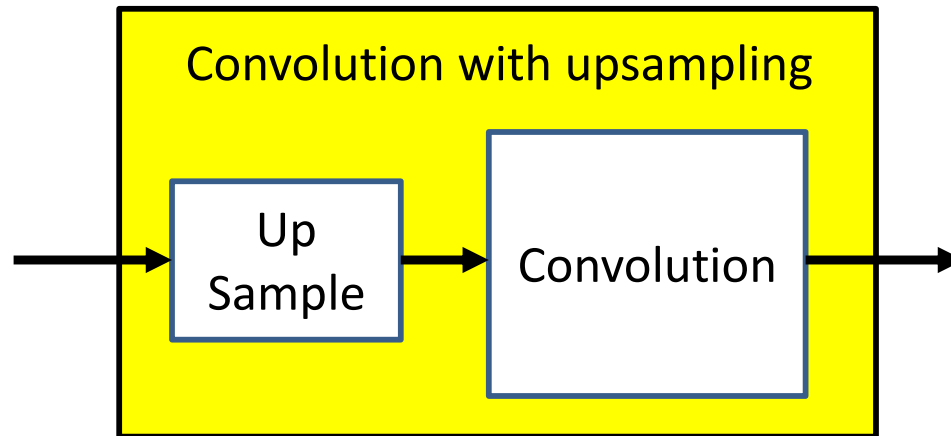
```
for j = 1:D1
    m = 1
    for x = 1:stride(1):W1-1-K1+1
        n = 1
        for y = 1:stride(1):H1-1-K1+1
            pidx(l,j,m,n) = maxidx(Y(l-1,j,x:x+K1-1,y:y+K1-1))
            Y(l,j,m,n) = Y(l-1,j,pidx(l,j,m,n))
            n = n+1
        end
        m = m+1
    end
end
```

Mean Pooling with Downsampling

Mean pooling

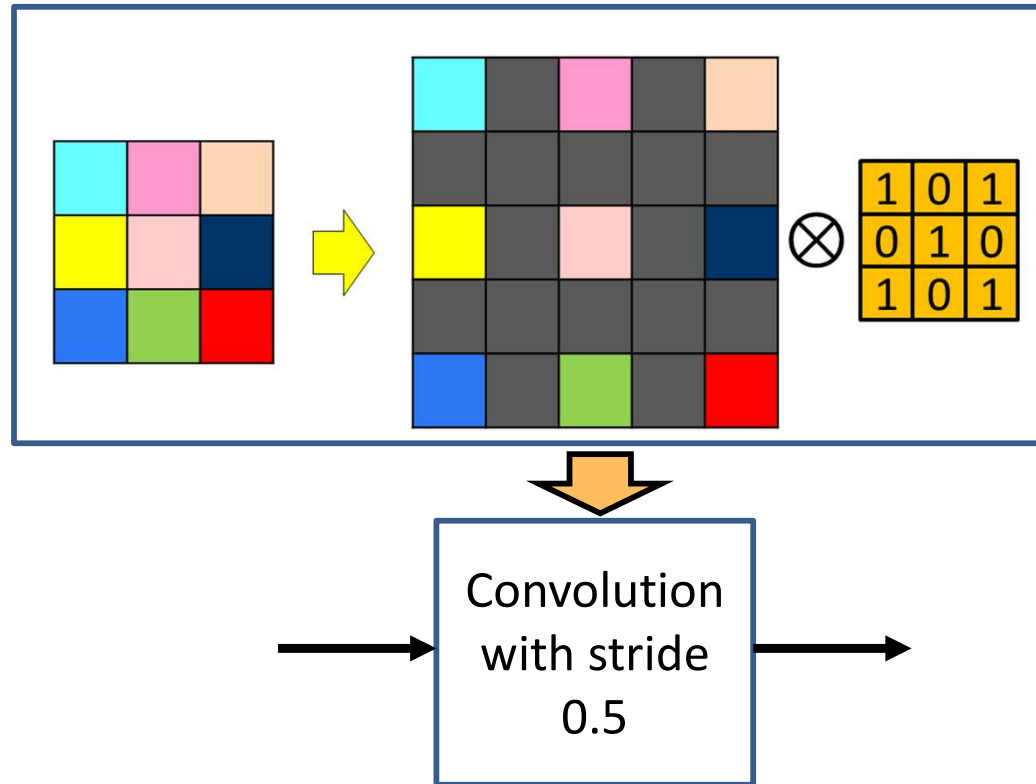
```
for j = 1:D1
    m = 1
    for x = 1:stride(1):W1-1-K1+1
        n = 1
        for y = 1:stride(1):H1-1-K1+1
            Y(l,j,m,n) = mean(Y(l-1,j,x:x+K1-1,y:y+K1-1))
            n = n+1
        end
        m = m+1
    end
end
```

The Upsampling Layer



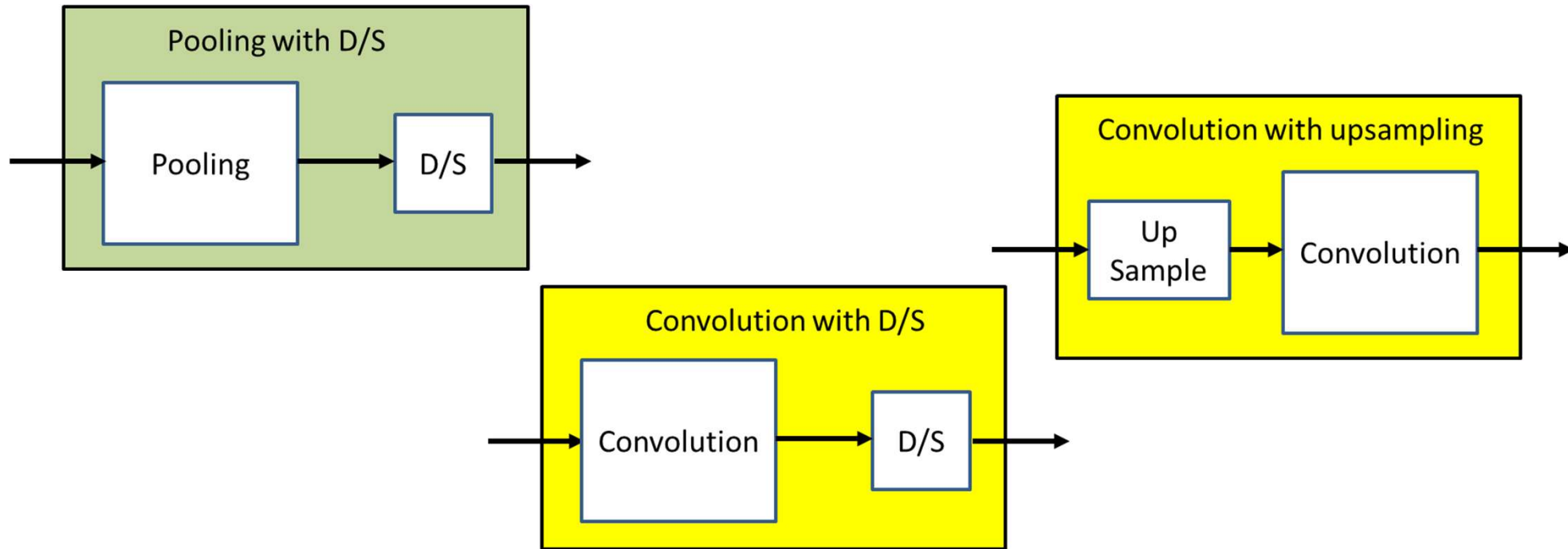
- A *upsampling* layer is generally followed by a CNN layer
 - It is not useful to follow it by a pooling layer
 - It is also not useful as the *final* layer of a CNN

The Upsampling Layer



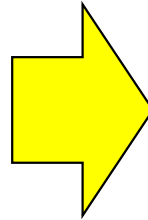
- Upsampling layers followed by a convolutional layer are also often viewed as convolving with a fractional stride
 - Upsampling by factor S is the same as striding by factor $1/S$
- Also called “transpose convolutions” for reasons we won’t get into here

* with resampling



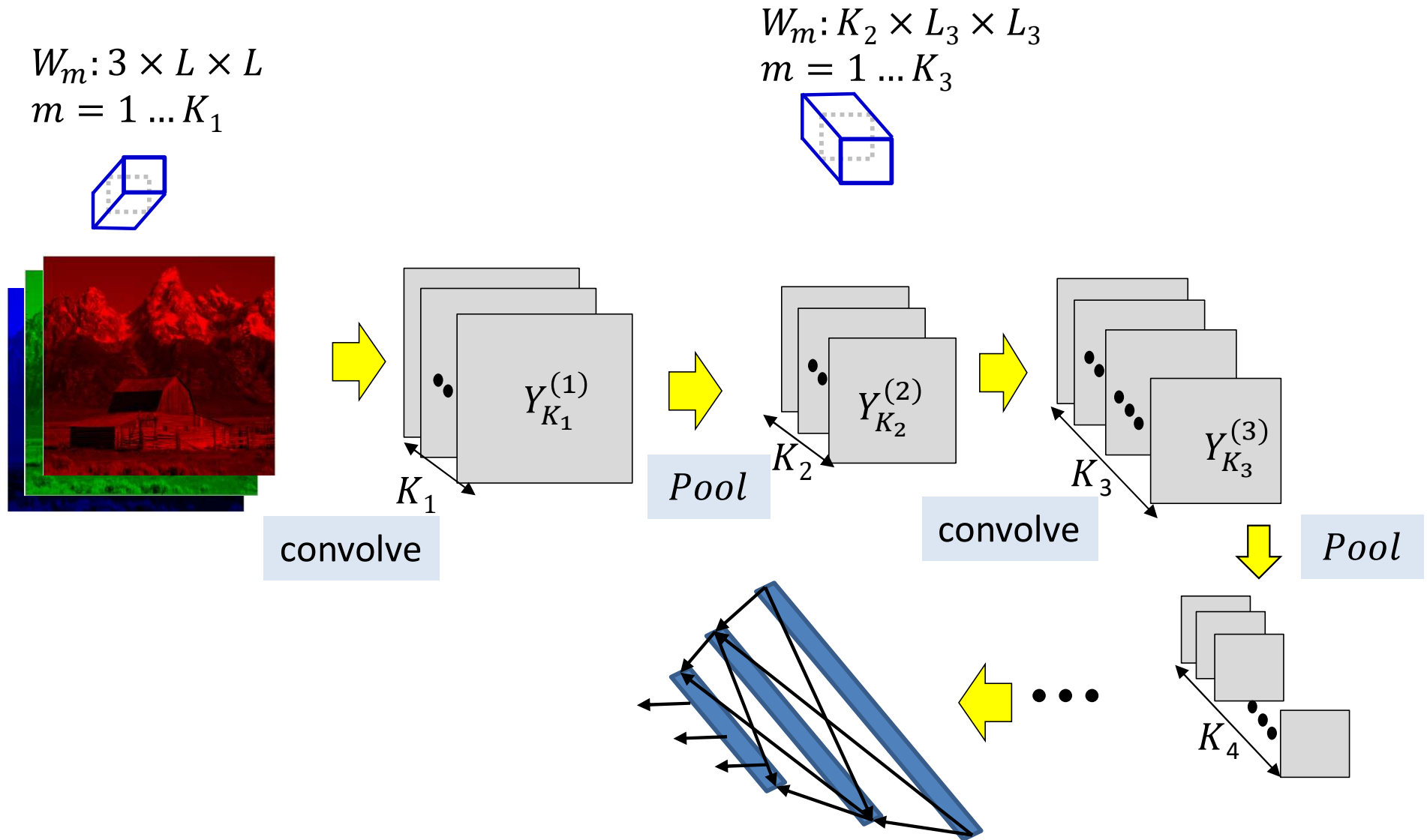
- Although the resampling operation is generally merged with convolutions or pooling (by changing their stride) in the forward pass in practical implementations...
- ...It is more convenient to think of the two as separate operations in the backward pass
 - More on this later...

Recap: A CNN, end-to-end



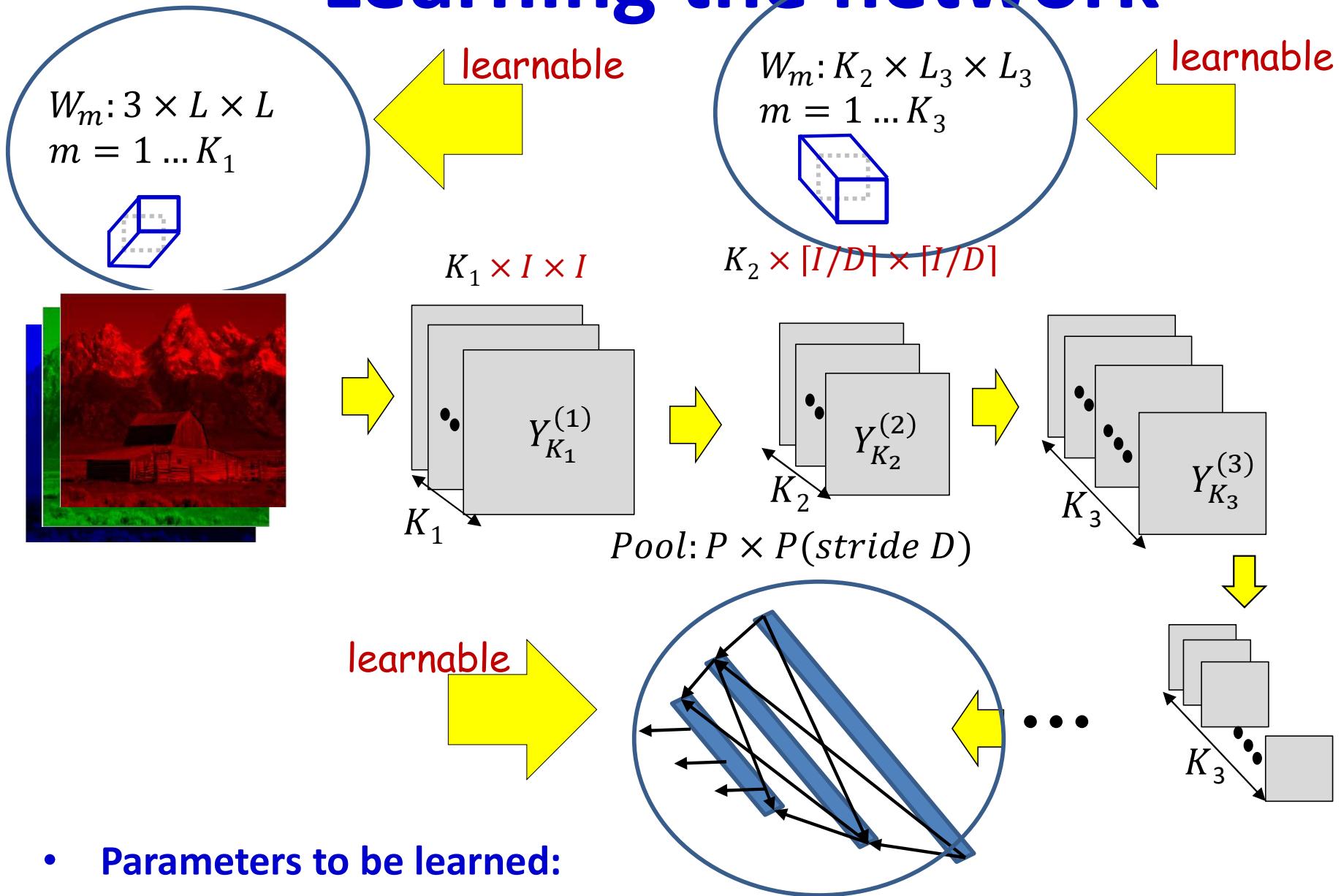
- Typical image classification task
 - Assuming maxpooling..
- Input: RGB images
 - Will assume color to be generic

Recap: A CNN, end-to-end



- Several convolutional and pooling layers.
- The output of the last layer is “flattened” and passed through an MLP

Learning the network

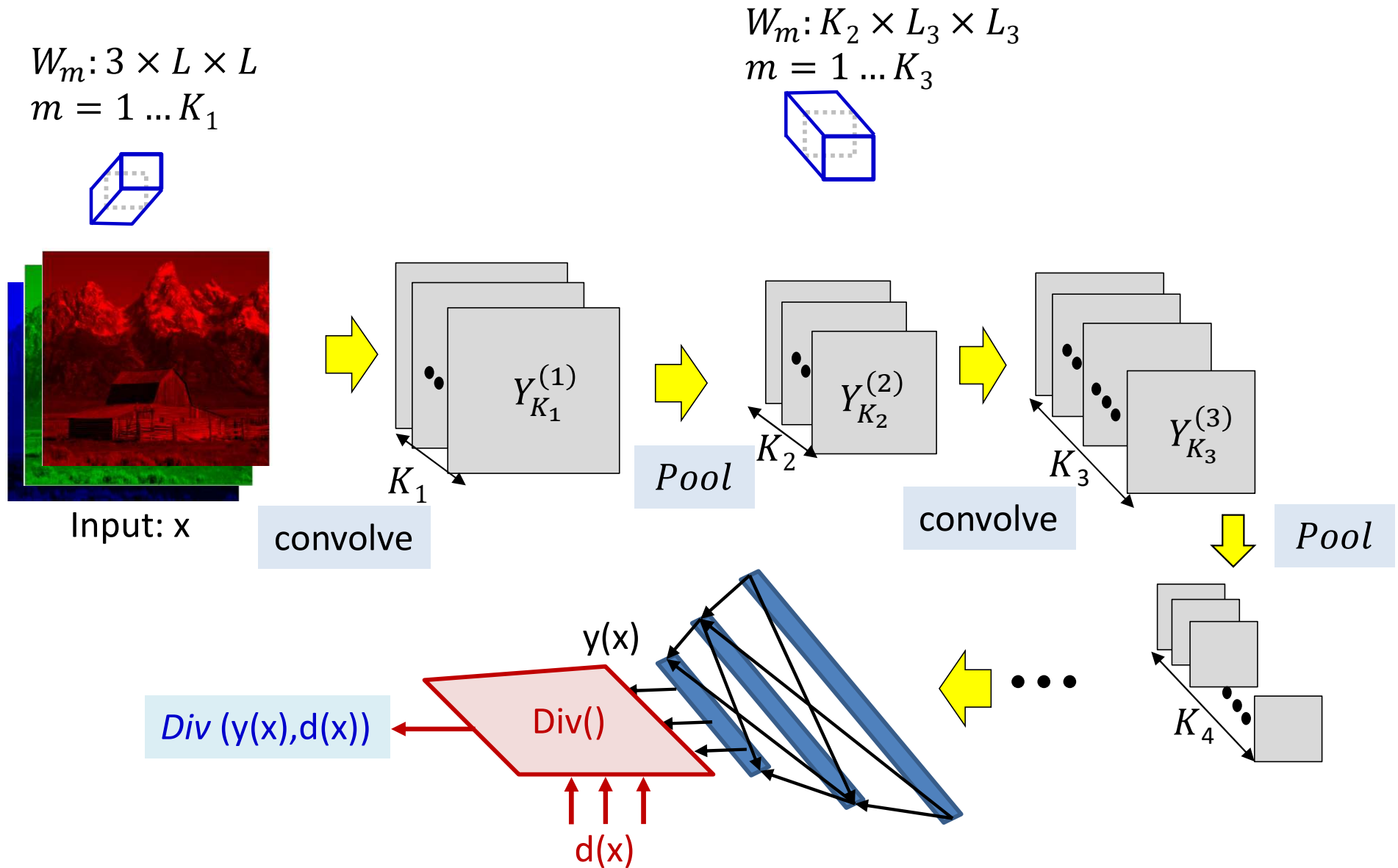


- Parameters to be learned:
 - The weights of the neurons in the final MLP
 - The (weights and biases of the) filters for every *convolutional* layer

Learning the CNN

- Training is as in the case of the regular MLP
 - The *only* difference is in the *structure* of the network
- **Training examples of (Image, class) are provided**
- **Define a loss:**
 - Define a divergence between the desired output and true output of the network in response to any input
 - The loss aggregates the divergences of the training set
- **Network parameters are trained to minimize the loss**
 - Through variants of gradient descent
 - Gradients are computed through backpropagation

Defining the loss



- The loss for a single instance

Problem Setup

- Given a training set of input-output pairs $(X_1, d_1), (X_2, d_2), \dots, (X_T, d_T)$
- The divergence on the i^{th} instance is $\text{div}(Y_i, d_i)$
- The aggregate Loss

$$\text{Loss} = \frac{1}{T} \sum_{i=1}^T \text{div}(Y_i, d_i)$$

- Minimize Loss w.r.t $\{W_m, b_m\}$
 - Using gradient descent

The derivative

Total training loss:

$$Loss = \frac{1}{T} \sum_i Div(Y_i, d_i)$$

- Computing the derivative

Total derivative:

$$\frac{dLoss}{dw} = \frac{1}{T} \sum_i \frac{dDiv(Y_i, d_i)}{dw}$$

The derivative

Total training loss:

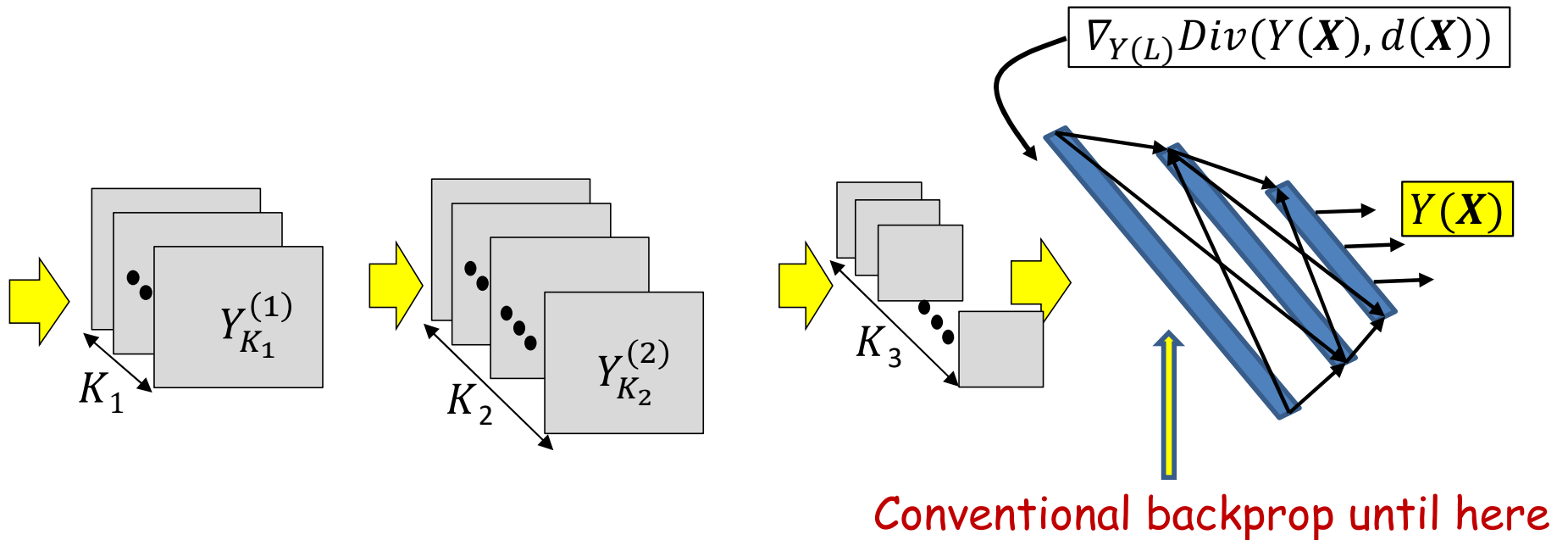
$$Loss = \frac{1}{T} \sum_i Div(Y_i, d_i)$$

- Computing the derivative

Total derivative:

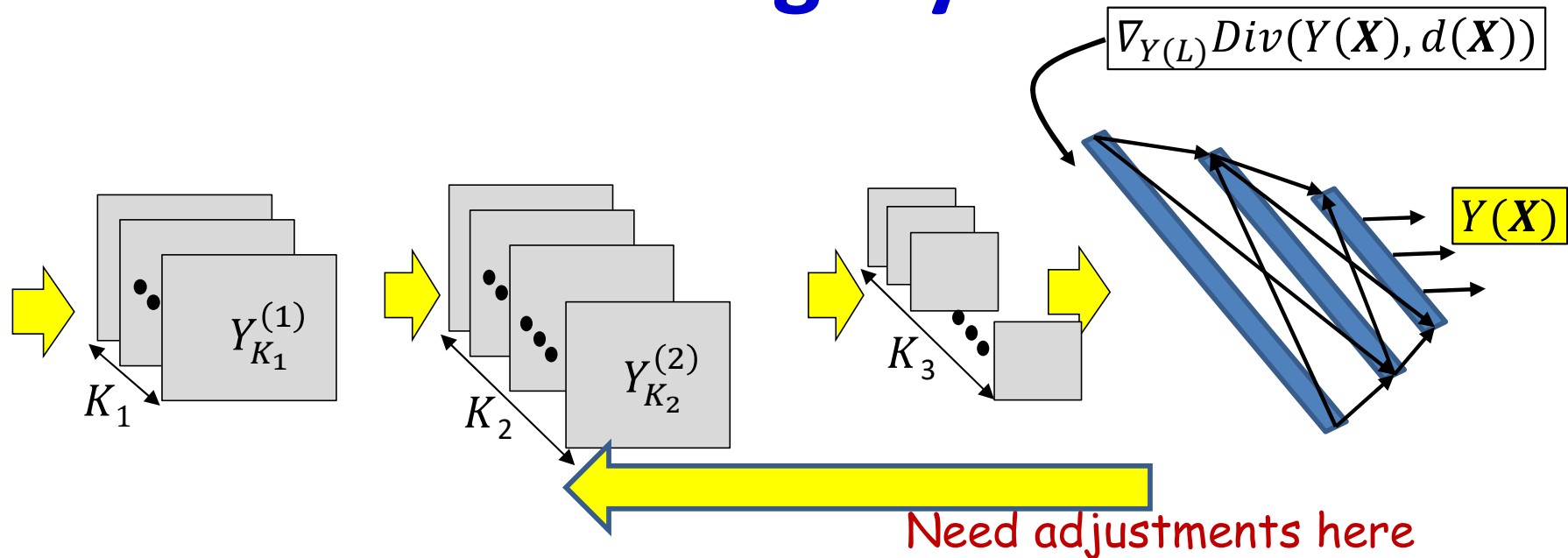
$$\frac{dLoss}{dw} = \frac{1}{T} \sum_i \frac{dDiv(Y_i, d_i)}{dw}$$

Backpropagation: Final flat layers



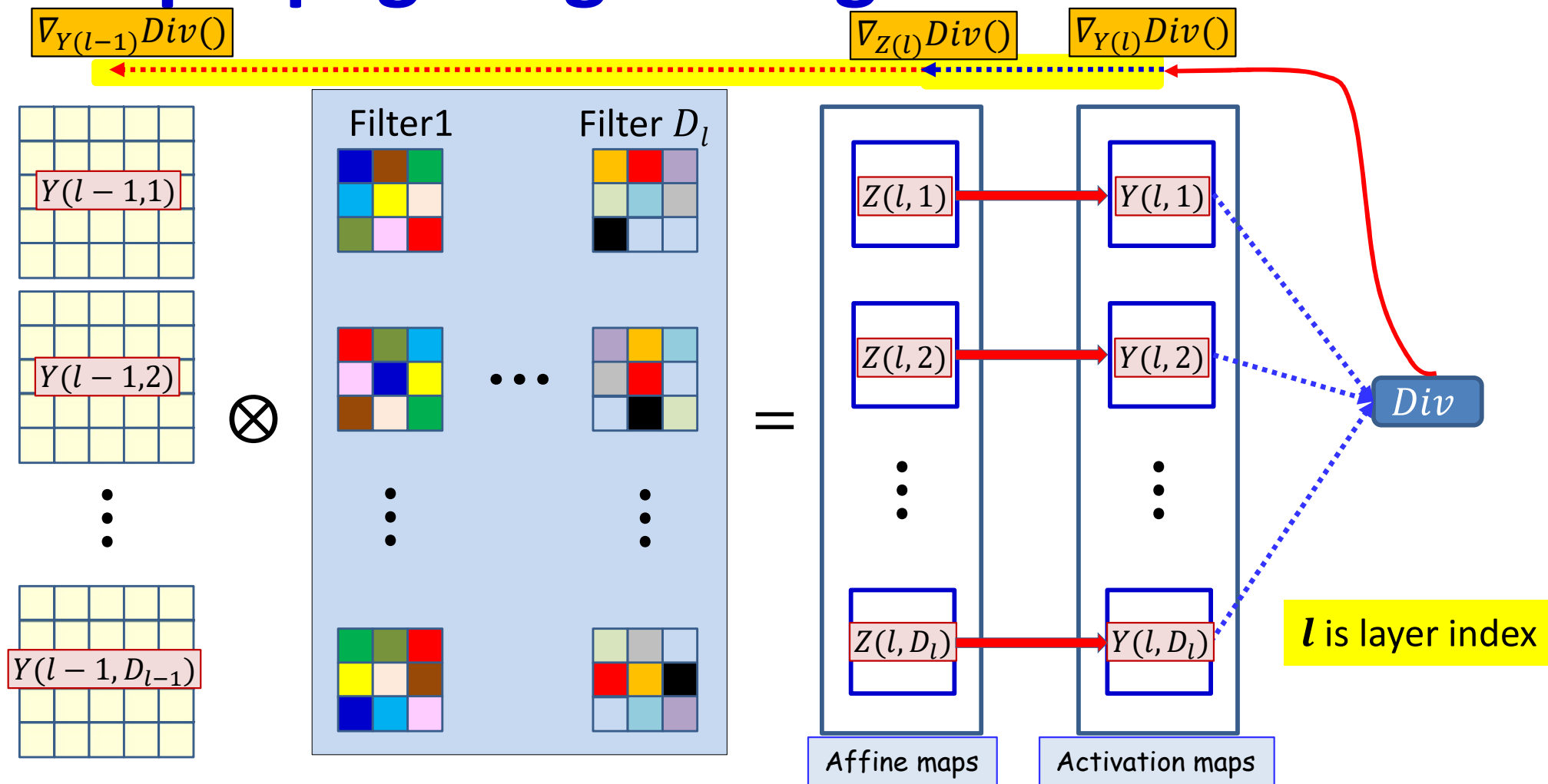
- For each training instance:
 - First, perform a forward pass through the net
 - Then perform the backpropagation of the derivative of the divergence
- Backpropagation continues in the usual manner until the computation of the derivative of the divergence w.r.t the inputs to the first “flat” layer
 - Important to recall: the first flat layer is only the “unrolling” of the maps from the final convolutional layer

Backpropagation: Convolutional and Pooling layers



- Backpropagation from the flat MLP requires special consideration of
 - The shared computation in the convolution layers
 - The pooling layers

Backpropagating through the convolution

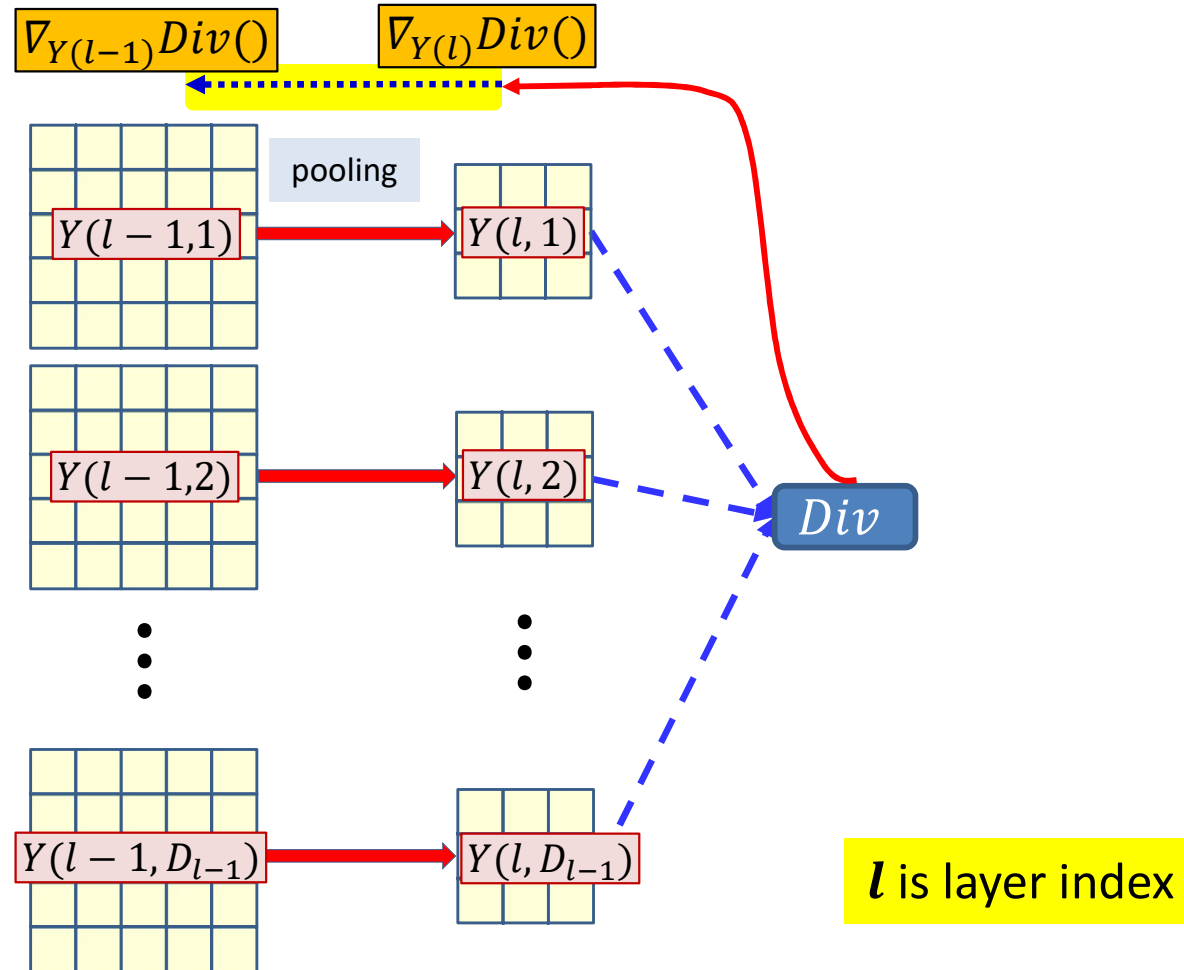


- **Convolution layers:**
- We already have the derivative w.r.t (all the elements of) activation map $Y(l,*)$
 - Having backpropagated it from the divergence
- We must backpropagate it through the activation to compute the derivative w.r.t. $Z(l,*)$ and further back to compute the derivative w.r.t the filters and $Y(l-1,*)$

Backpropagation: Convolutional layers

- **Assumption:** We already have the derivatives w.r.t. the elements of the maps output by the final convolutional (or pooling) layer
 - Obtained as a result of backpropagating through the flat MLP
- **Required:**
 - **For convolutional layers:**
 - How to compute the derivatives w.r.t. the affine combination $Z(l)$ maps from the activation output maps $Y(l)$
 - How to compute the derivative w.r.t. $Y(l - 1)$ and $w(l)$ given derivatives w.r.t. $Z(l)$

Backprop: Pooling layer



- **Pooling layers:**
- We already have the derivative w.r.t $Y(l,*)$
 - Having backpropagated it from the divergence
- We must compute the derivative w.r.t $Y(l-1,*)$

Backpropagation: Convolutional and Pooling layers

- **Assumption:** We already have the derivatives w.r.t. the elements of the maps output by the final convolutional (or pooling) layer
 - Obtained as a result of backpropagating through the flat MLP
- **Required:**
 - **For convolutional layers:**
 - How to compute the derivatives w.r.t. the affine combination $Z(l)$ maps from the activation output maps $Y(l)$
 - How to compute the derivative w.r.t. $Y(l - 1)$ and $w(l)$ given derivatives w.r.t. $Z(l)$
 - **For pooling layers:**
 - How to compute the derivative w.r.t. $Y(l - 1)$ given derivatives w.r.t. $Y(l)$

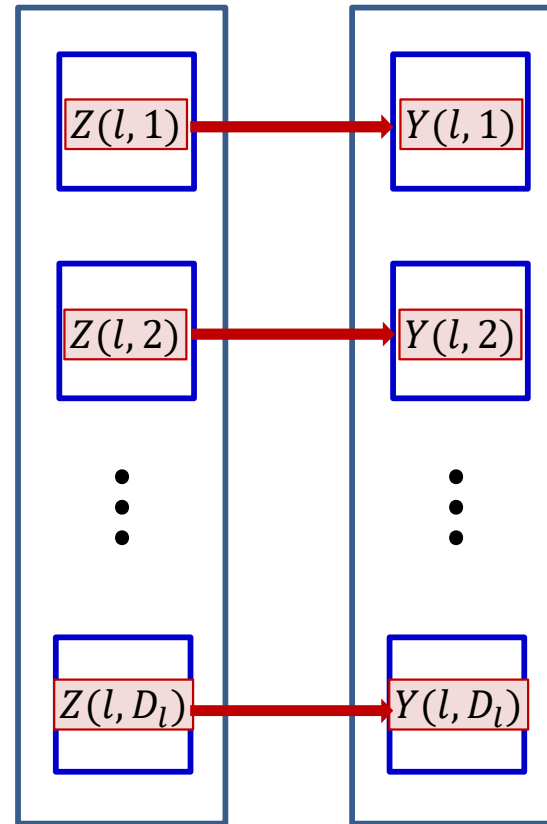
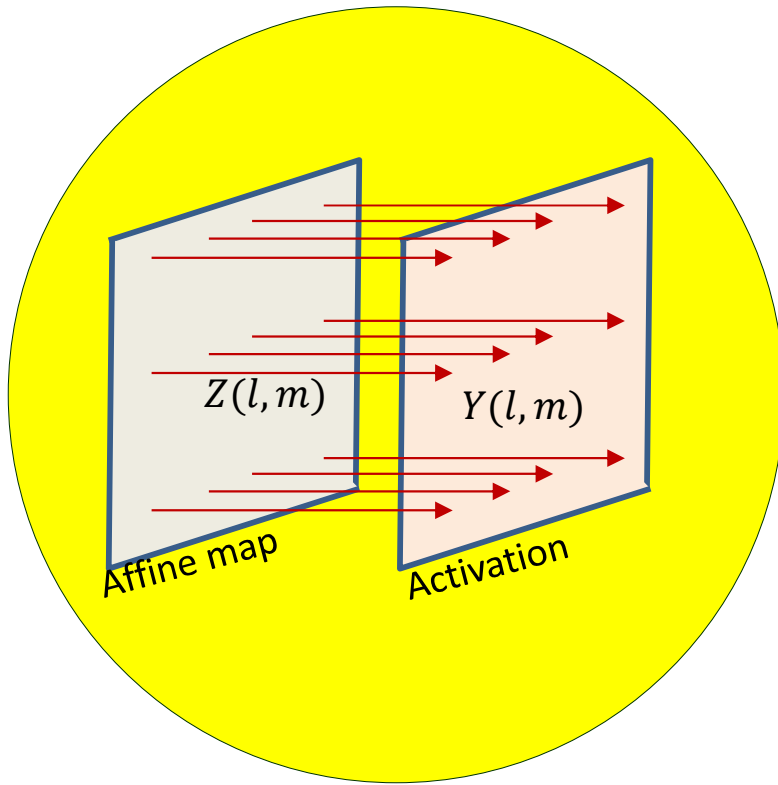
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 - **For pooling layers:**
 - How to compute the derivative w.r.t. $Y(l - 1)$ given derivatives w.r.t. $Y(l)$

Backpropagation: Convolutional and Pooling layers

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Backpropagating through the activation

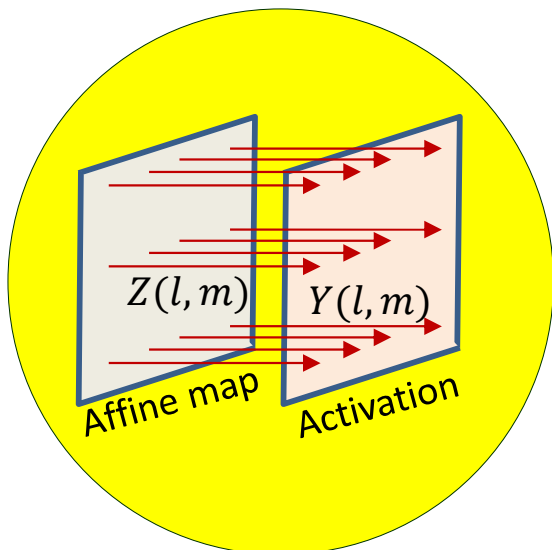


- **Forward computation:** The activation maps are obtained by point-wise application of the activation function to the affine maps

$$y(l, m, x, y) = f(z(l, m, x, y))$$

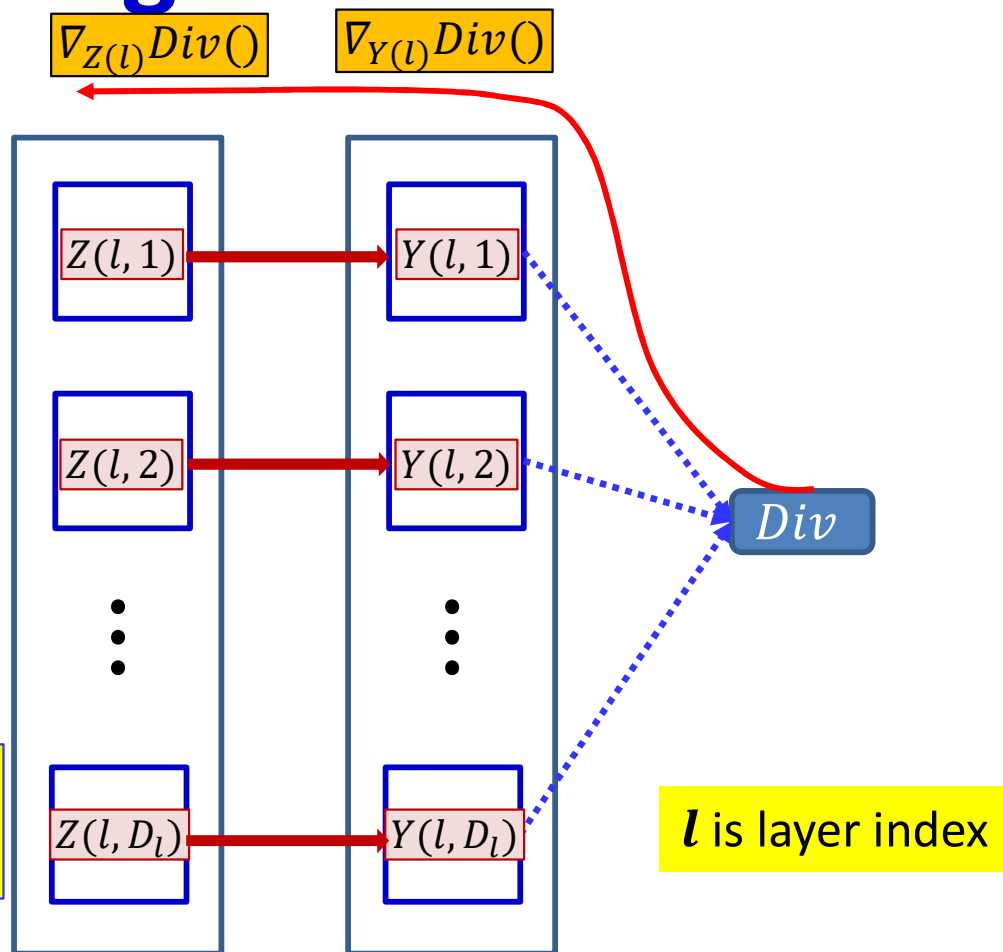
- The affine map entries $z(l, m, x, y)$ have already been computed via convolutions over the previous layer

Backpropagating through the activation



$$y(l, m, x, y) = f(z(l, m, x, y))$$

$$\frac{dDiv}{dz(l, m, x, y)} = \frac{dDiv}{dy(l, m, x, y)} f'(z(l, m, x, y))$$



- **Backward computation:** For every map $Y(l, m)$ for every position (x, y) , we already have the derivative of the divergence w.r.t. $y(l, m, x, y)$
 - Obtained via backpropagation
- We obtain the derivatives of the divergence w.r.t. $z(l, m, x, y)$ using the chain rule:

$$\frac{dDiv}{dz(l, m, x, y)} = \frac{dDiv}{dy(l, m, x, y)} f'(z(l, m, x, y))$$

- Simple component-wise computation

Backpropagation: Convolutional and Pooling layers

- **Assumption:** We already have the derivatives w.r.t. the elements of the maps output by the final convolutional (or pooling) layer
 - Obtained as a result of backpropagating through the flat MLP
- **Required:**
 - **For convolutional layers:**
 - ✓ How to compute the derivatives w.r.t. the affine combination $Z(l)$ maps from the activation output maps $Y(l)$
 - How to compute the derivative w.r.t. $Y(l - 1)$ and $w(l)$ given derivatives w.r.t. $Z(l)$
 - **For pooling layers:**
 - How to compute the derivative w.r.t. $Y(l - 1)$ given derivatives w.r.t. $Y(l)$

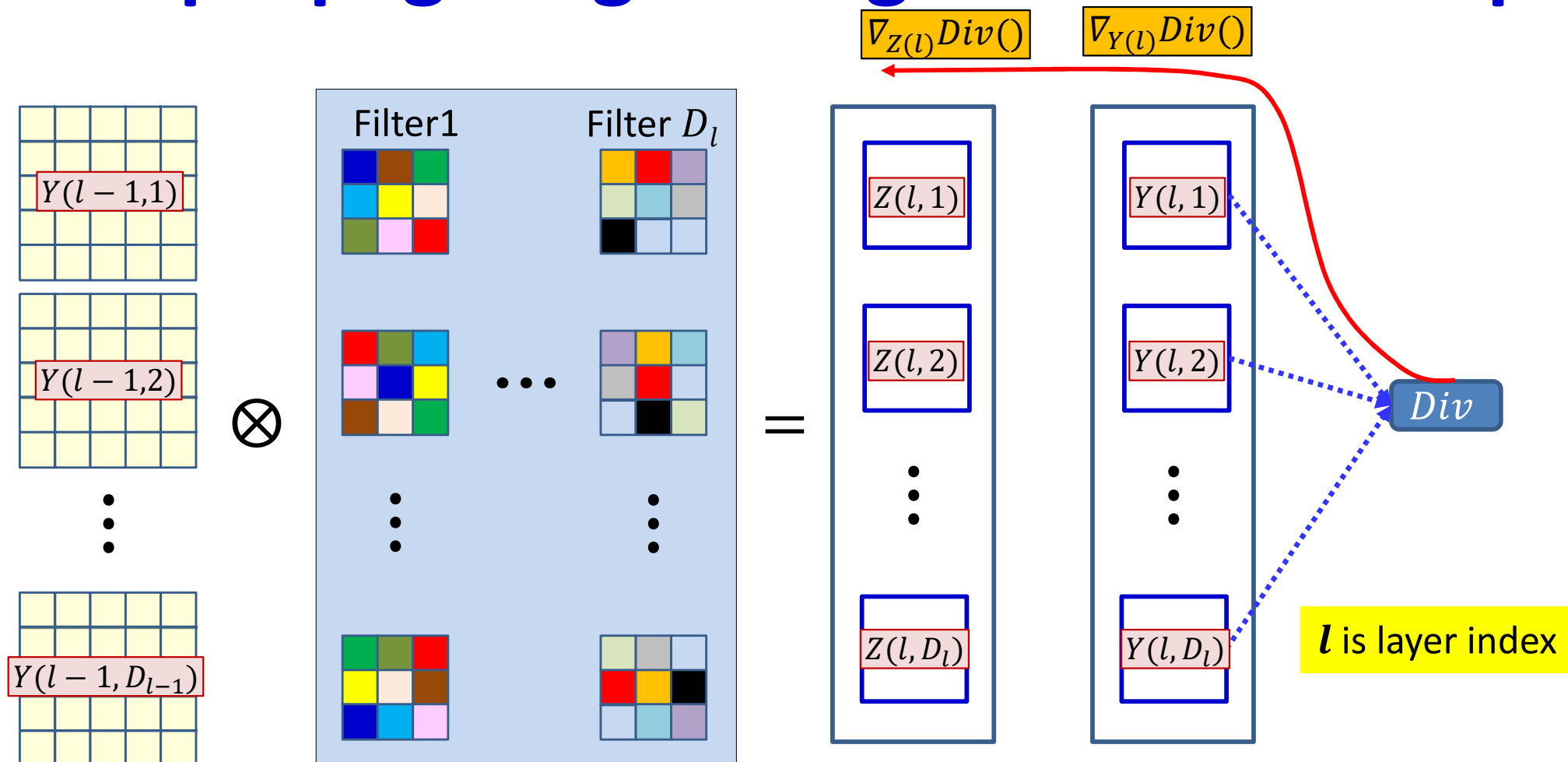
Backpropagating through affine map

- Forward affine computation:
 - Compute affine maps $z(l, n, x, y)$ from previous layer maps $y(l - 1, m, x, y)$ and filters $w_l(m, n, x, y)$
- Backpropagation: Given $\frac{dDiv}{dz(l, n, x, y)}$
 - Compute derivative w.r.t. $y(l - 1, m, x, y)$
 - Compute derivative w.r.t. $w_l(m, n, x, y)$

Backpropagating through affine map

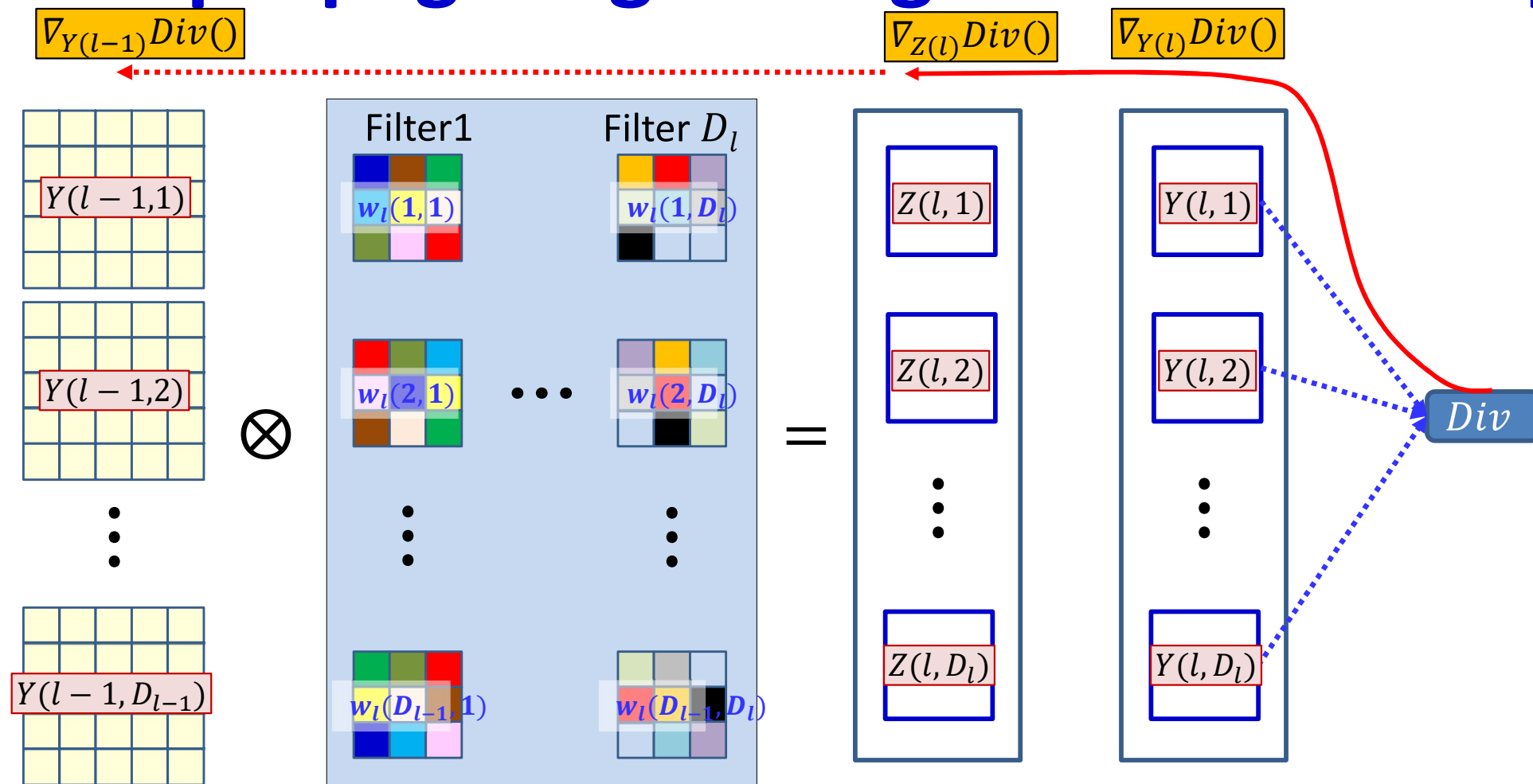
- Forward affine computation:
 - Compute affine maps $z(l, n, x, y)$ from previous layer maps $y(l - 1, m, x, y)$ and filters $w_l(m, n, x, y)$
- Backpropagation: Given $\frac{dDiv}{dz(l, n, x, y)}$
 - Compute derivative w.r.t. $y(l - 1, m, x, y)$
 - Compute derivative w.r.t. $w_l(m, n, x, y)$

Backpropagating through the affine map



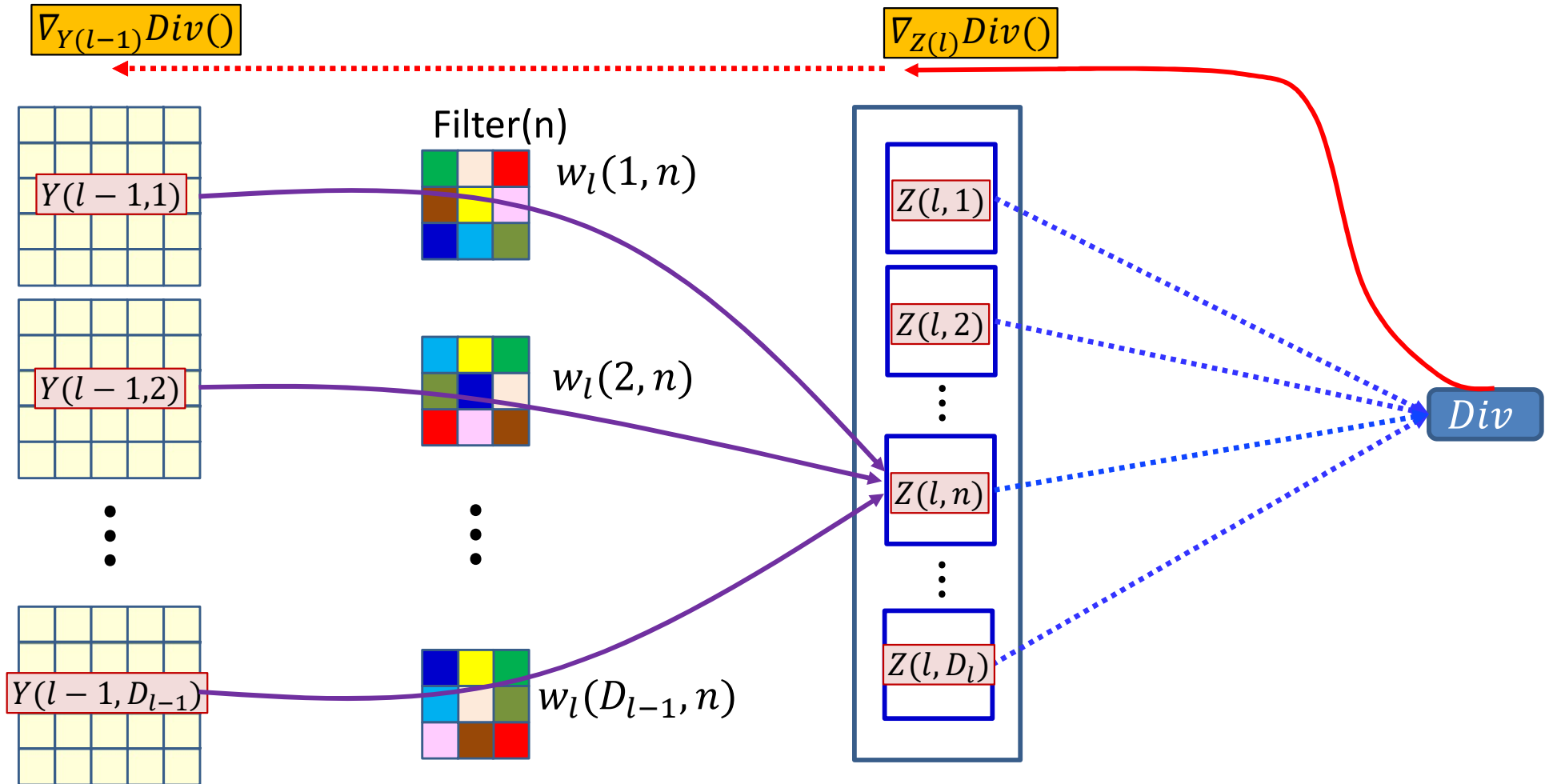
- We already have the derivative w.r.t $Z(l,*)$
 - Having backpropagated it past $Y(l,*)$

Backpropagating through the affine map



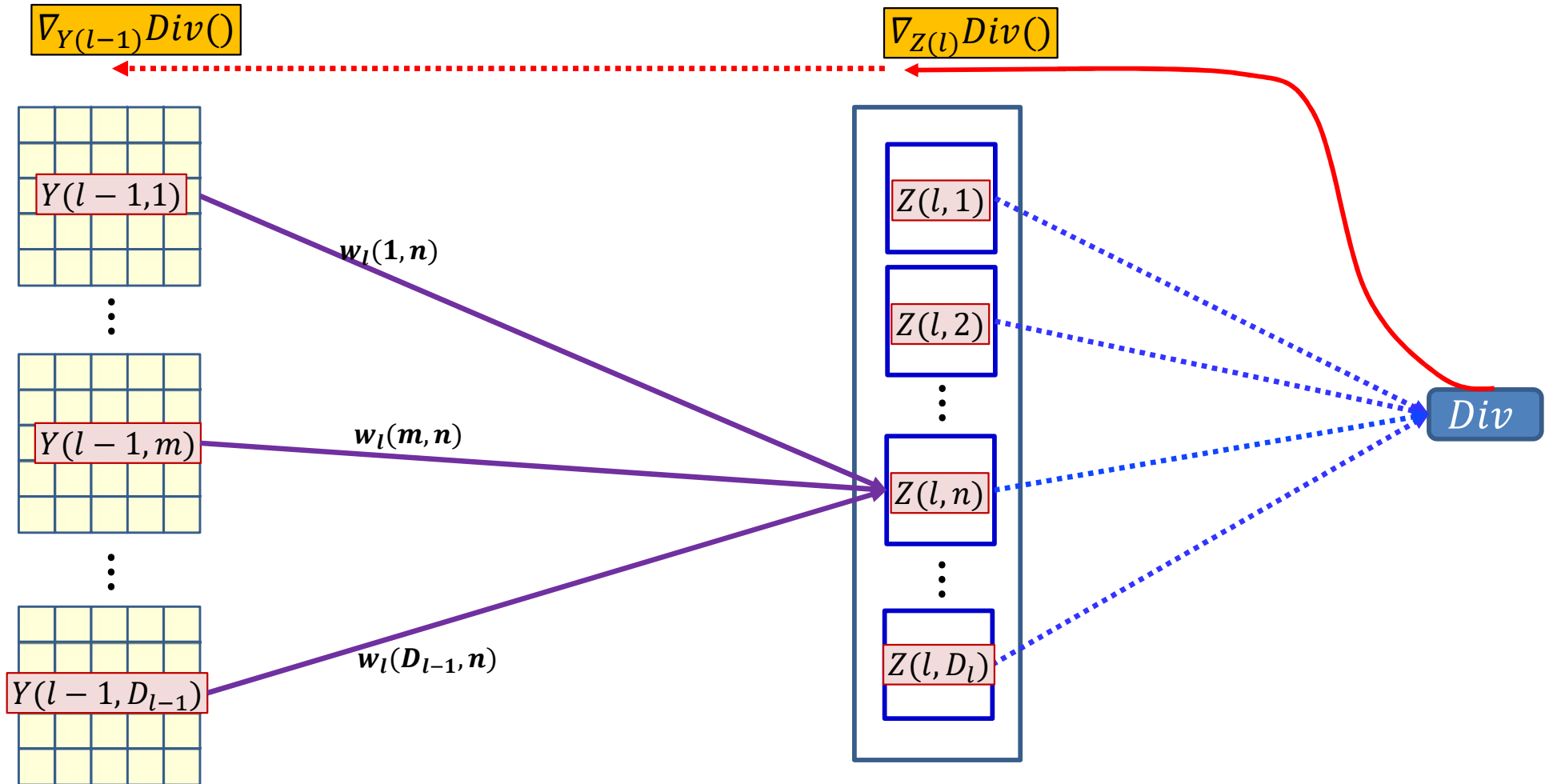
- We already have the derivative w.r.t $Z(l,*)$
 - Having backpropagated it past $Y(l,*)$
- We must compute the derivative w.r.t $Y(l-1,*)$

Dependency between $Z(l,n)$ and $Y(l-1,*)$



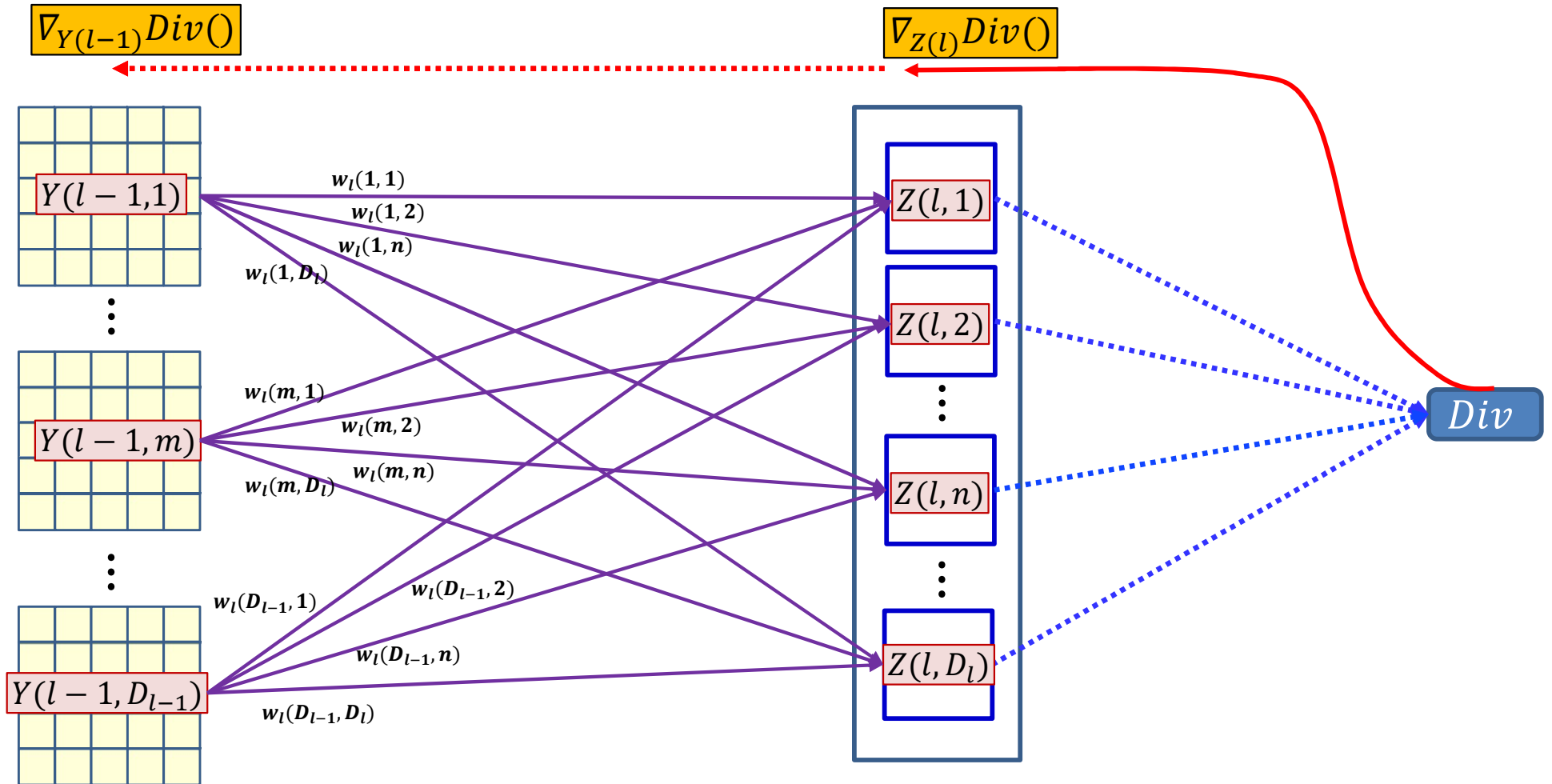
- Each $Y(l-1, m)$ map/channel influences $Z(l, n)$ through the m th channel of the n th filter $w_l(m, n)$

Dependency between $Z(l,n)$ and $Y(l-1,*)$



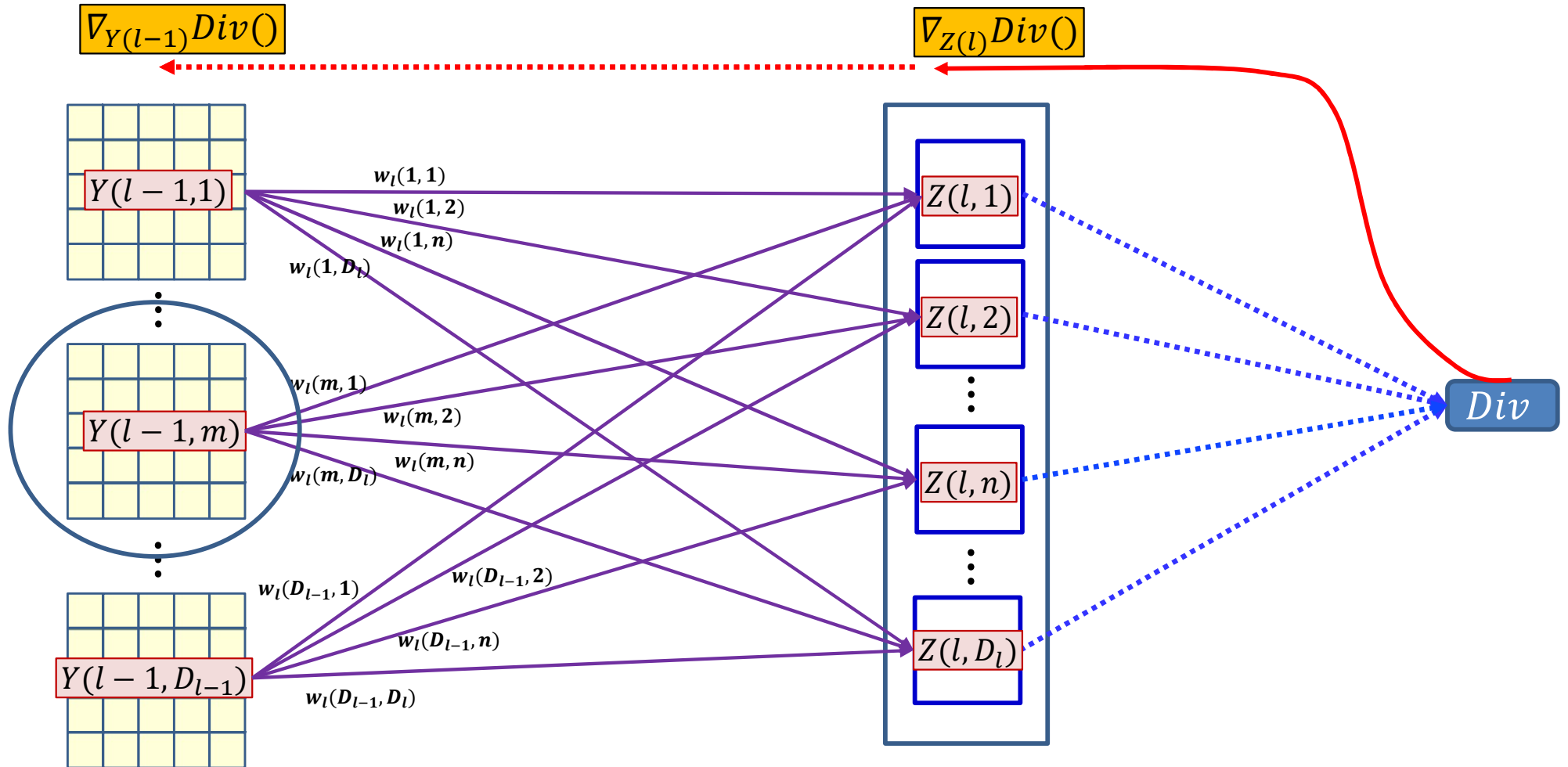
- Each $Y(l-1, m)$ map/channel influences $Z(l, n)$ through the m th channel of the n th filter $w_l(m, n)$

Dependency between $Z(l,*)$ and $Y(l-1,*)$



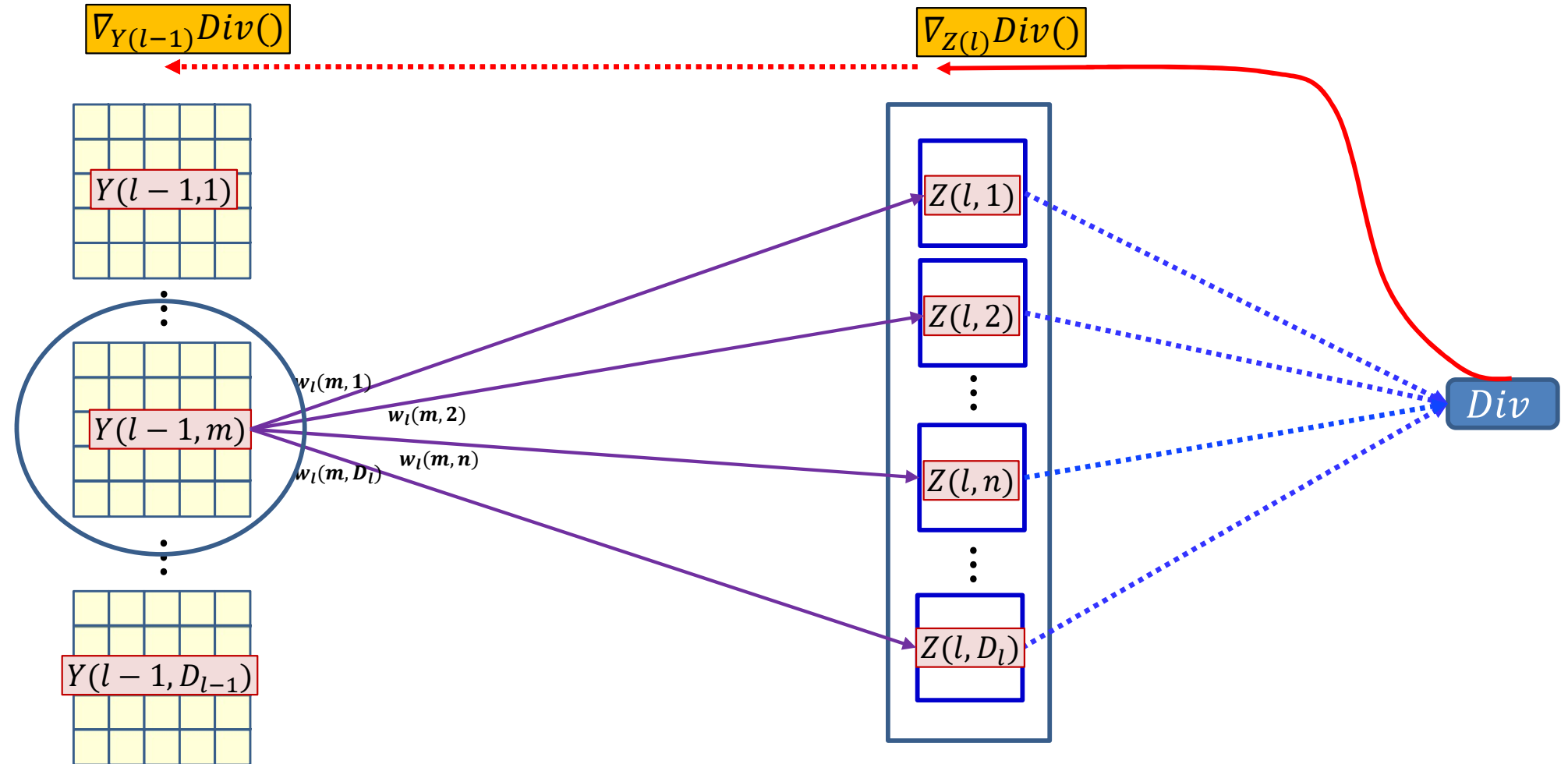
- Each $Y(l-1, m)$ map/channel influences $Z(l, n)$ through the m th channel of the n th filter $w_l(m, n)$

Dependency between $Z(l,*)$ and $Y(l-1,*)$



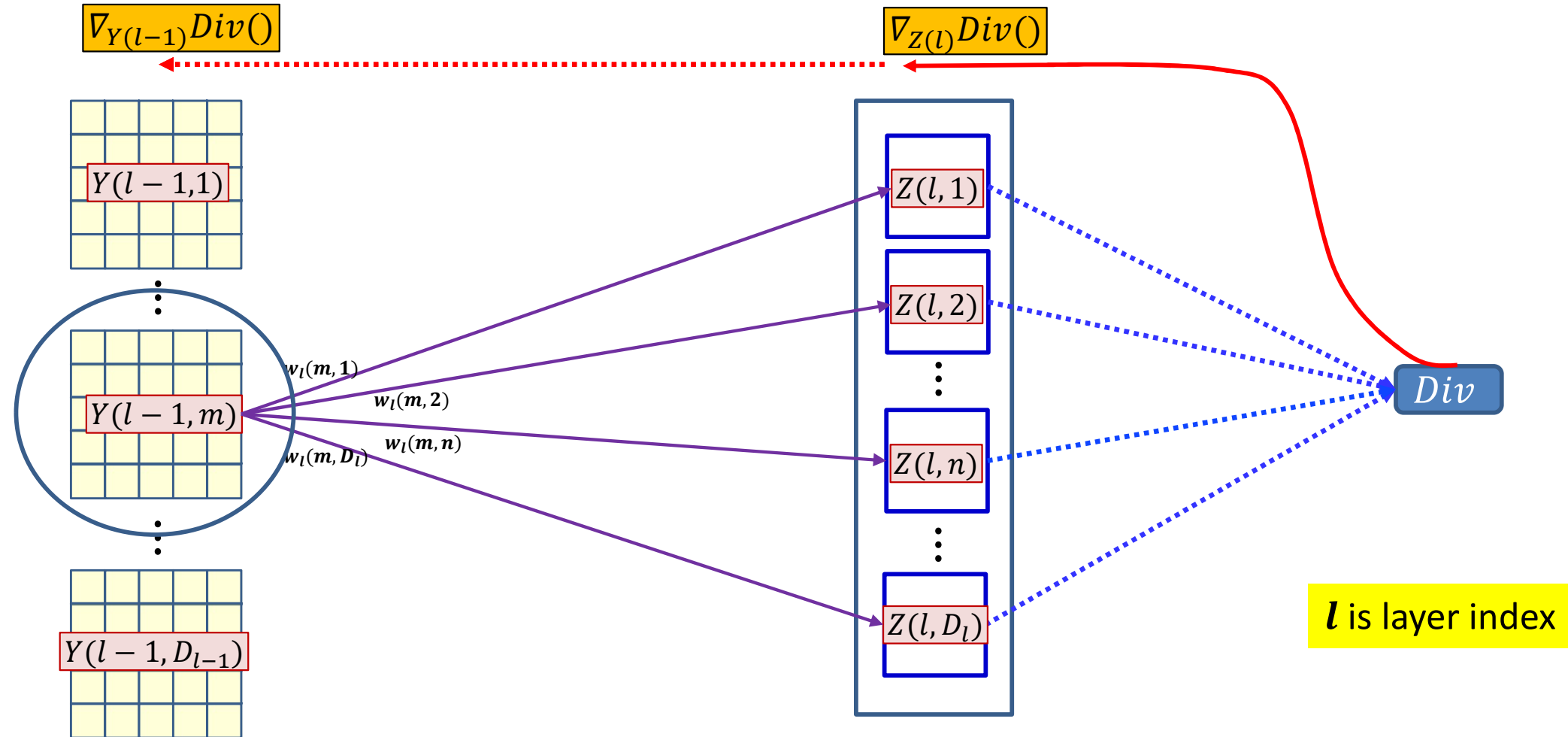
- Each $Y(l-1, m)$ map/channel influences $Z(l, n)$ through the m th channel of the n th filter $w_l(m, n)$

Dependency diagram for a single map



- Each $Y(l-1, m)$ map/channel influences $Z(l, n)$ through the m th channel of the n th filter $w_l(m, n)$
- $Y(l-1, m, *, *)$ influences the divergence through all $Z(l, n, *, *)$ maps

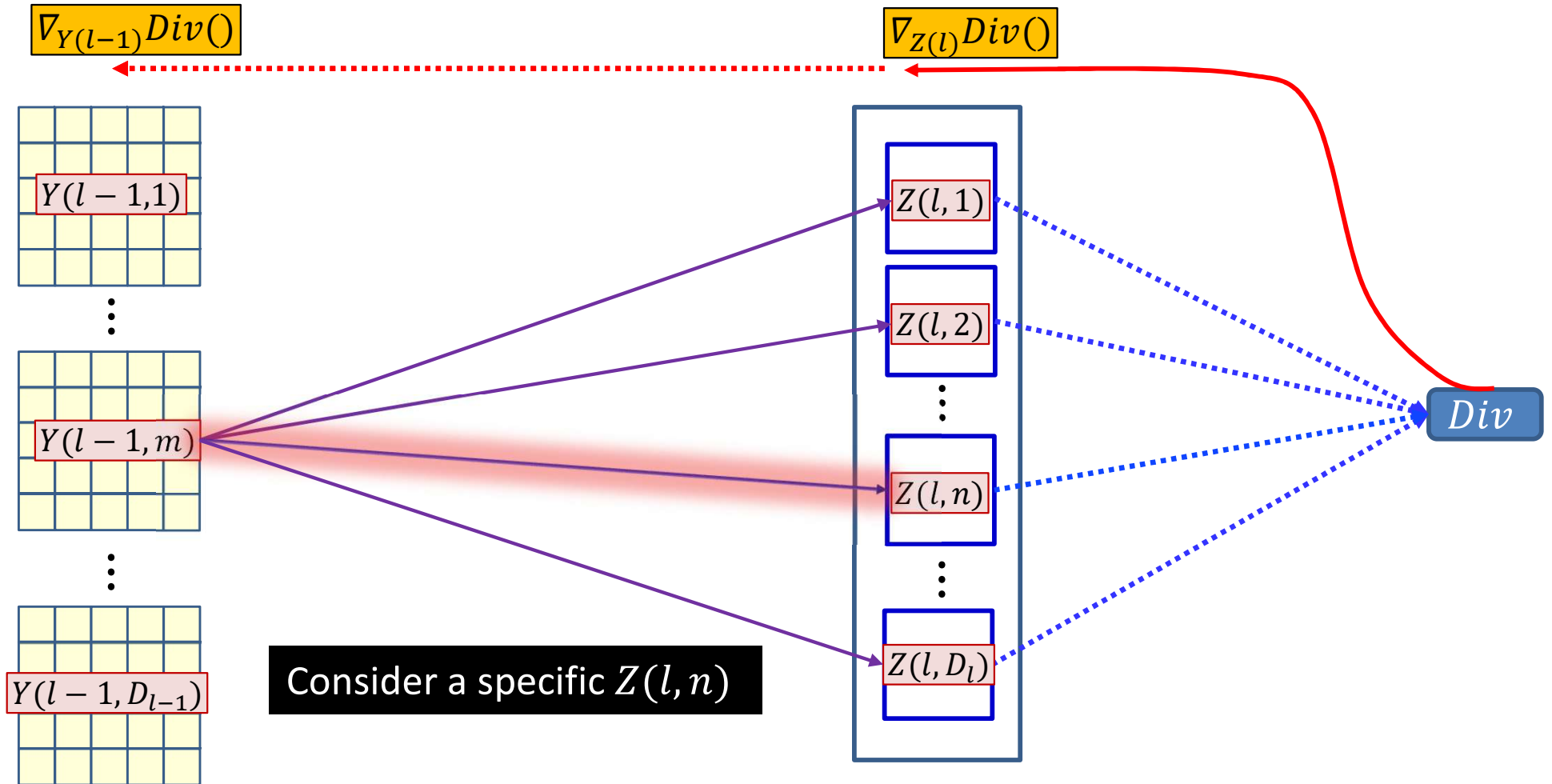
Dependency diagram for a single map



$$\nabla_{Y(l-1, m)} Div(.) = \sum_n \nabla_{Z(l, n)} Div(.) \underbrace{\nabla_{Y(l-1, m)} Z(l, n)}$$

- Need to compute $\nabla_{Y(l-1, m)} Z(l, n)$, the derivative of $Z(l, n)$ w.r.t. $Y(l-1, m)$ to complete the computation of the formula

Dependency diagram for a single map



$$\nabla_{Y(l-1,m)} Div(.) = \sum_n \nabla_{Z(l,n)} Div(.) \underbrace{\nabla_{Y(l-1,m)} Z(l,n)}$$

- Need to compute $\nabla_{Y(l-1,m)} Z(l,n)$, the derivative of $Z(l,n)$ w.r.t. $Y(l-1,m)$ to complete the computation of the formula

BP: Convolutional layer

1 _{x1}	1 _{x0}	1 _{x1}	0	0
0 _{x0}	1 _{x1}	1 _{x0}	1	0
0 _{x1}	0 _{x0}	1 _{x1}	1	1
0	0	1	1	0
0	1	1	0	0

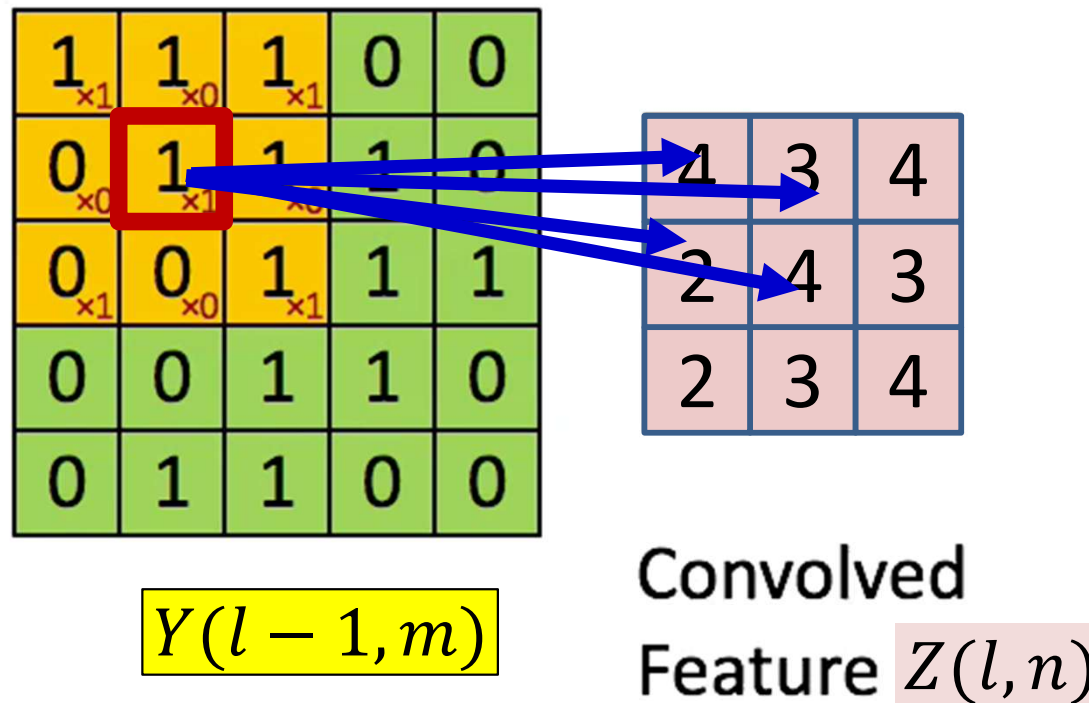
$Y(l-1, m)$

4		

Convolved
Feature $Z(l, n)$

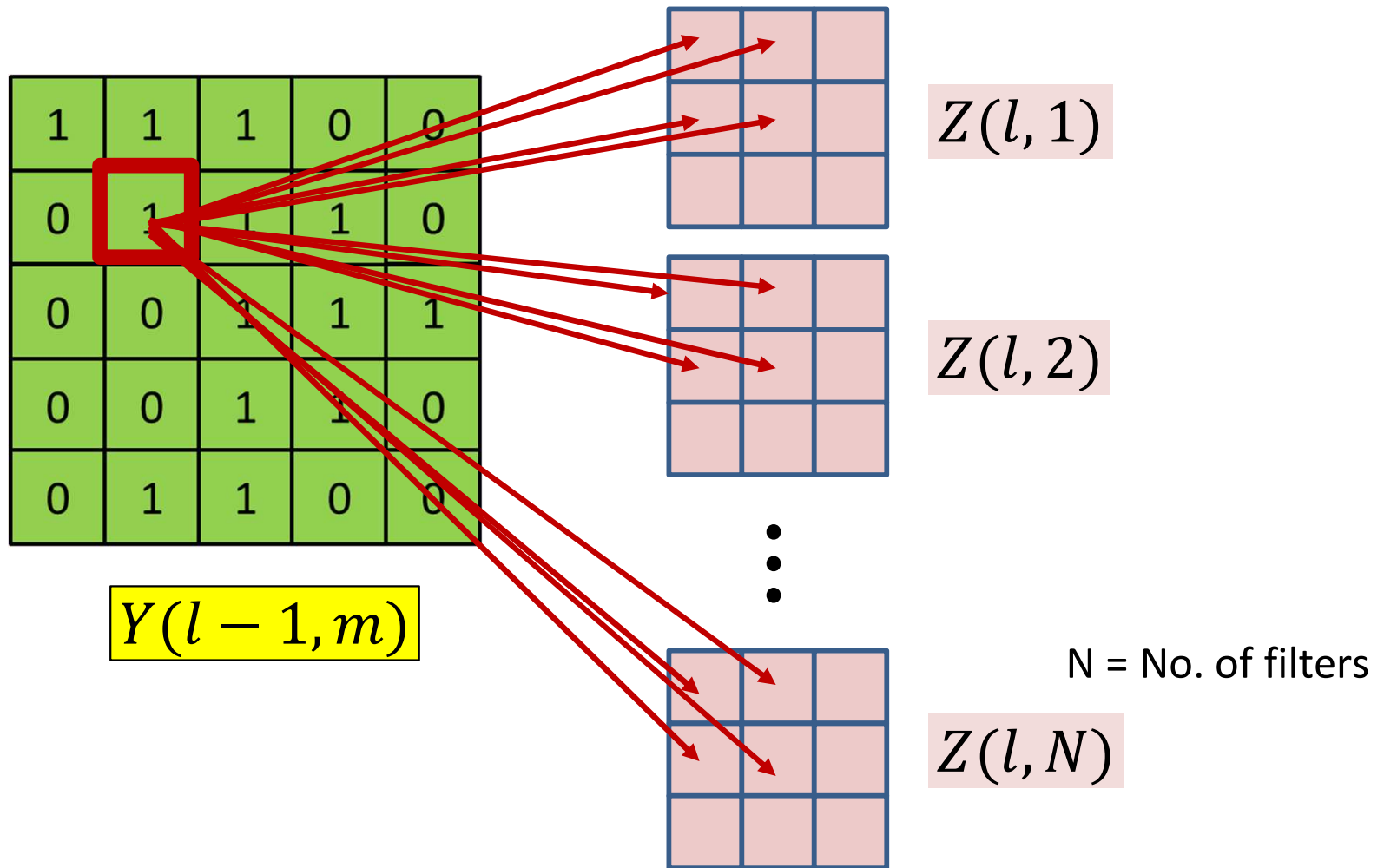
- Each $Y(l-1, m, x, y)$ affects several $z(l, n, x', y')$ terms

BP: Convolutional layer



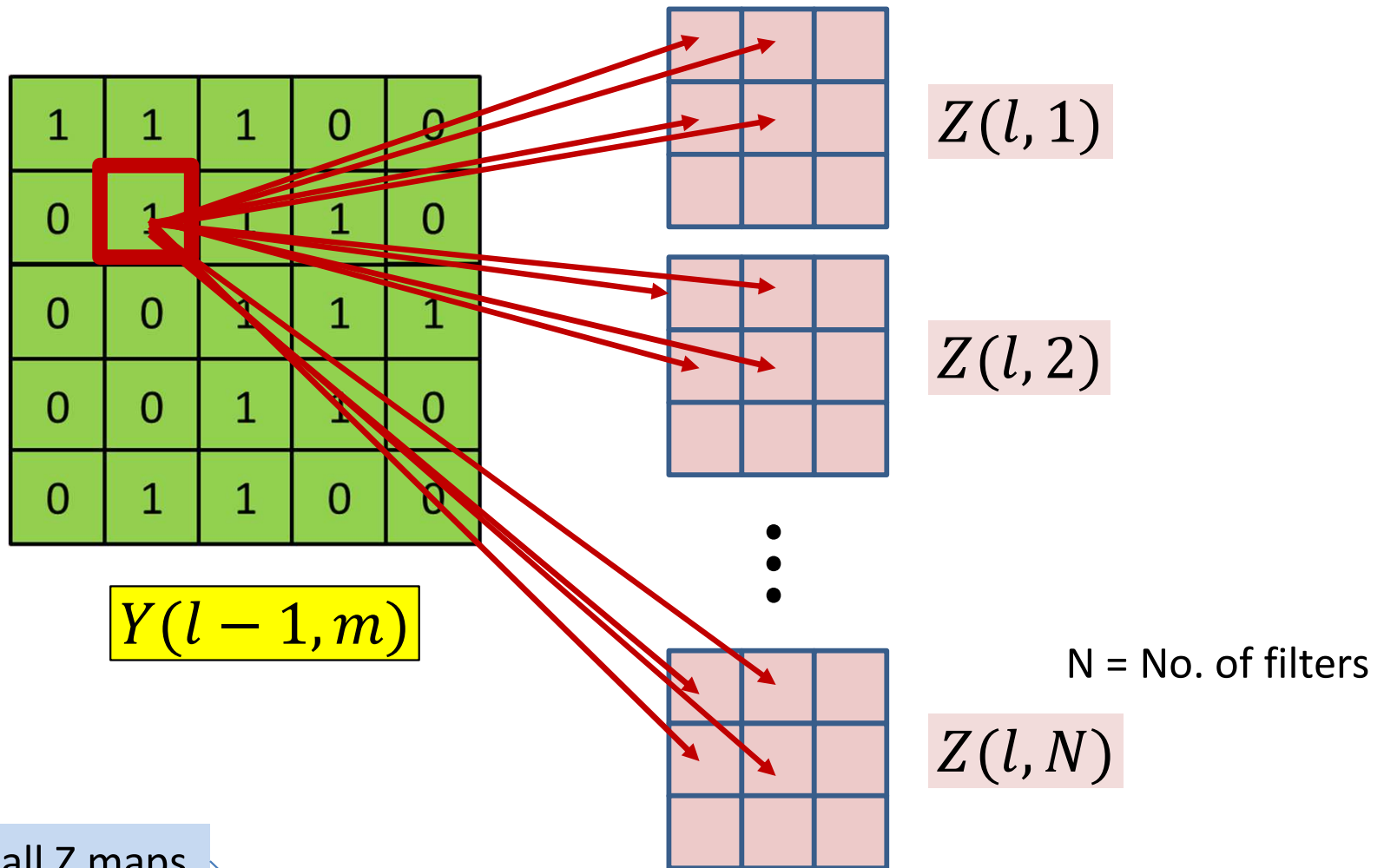
- Each $Y(l-1, m, x, y)$ affects several $z(l, n, x', y')$ terms

BP: Convolutional layer



- Each $Y(l-1, m, x, y)$ affects several $z(l, n, x', y')$ terms
 - Affects terms in *all* l^{th} layer Z maps

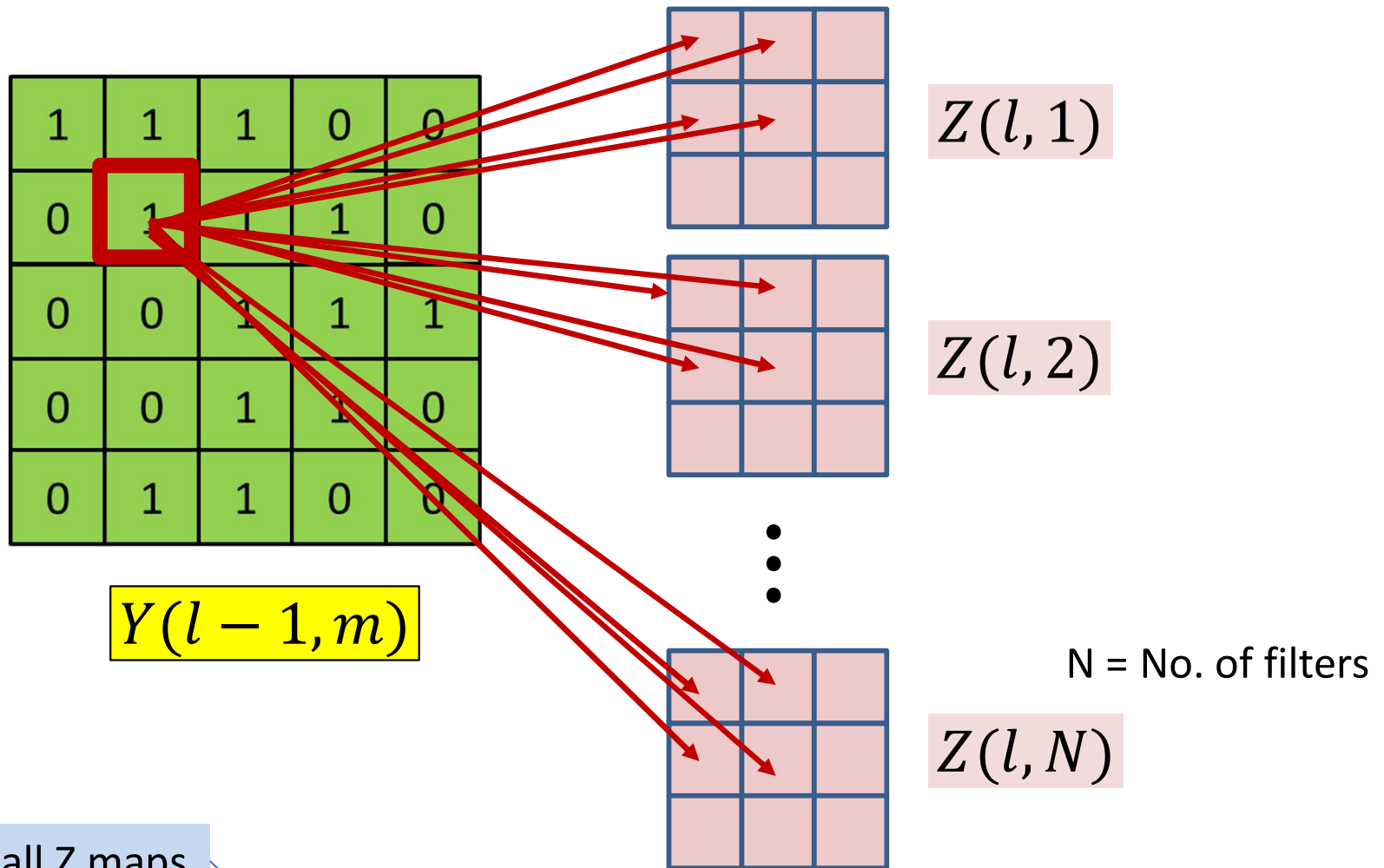
BP: Convolutional layer



Summing over all Z maps

$$\frac{dDiv}{dY(l-1, m, x, y)} = \sum_n \sum_{x', y'} \frac{dDiv}{dz(l, n, x', y')} \frac{dz(l, n, x', y')}{dY(l-1, m, x, y)}$$

BP: Convolutional layer

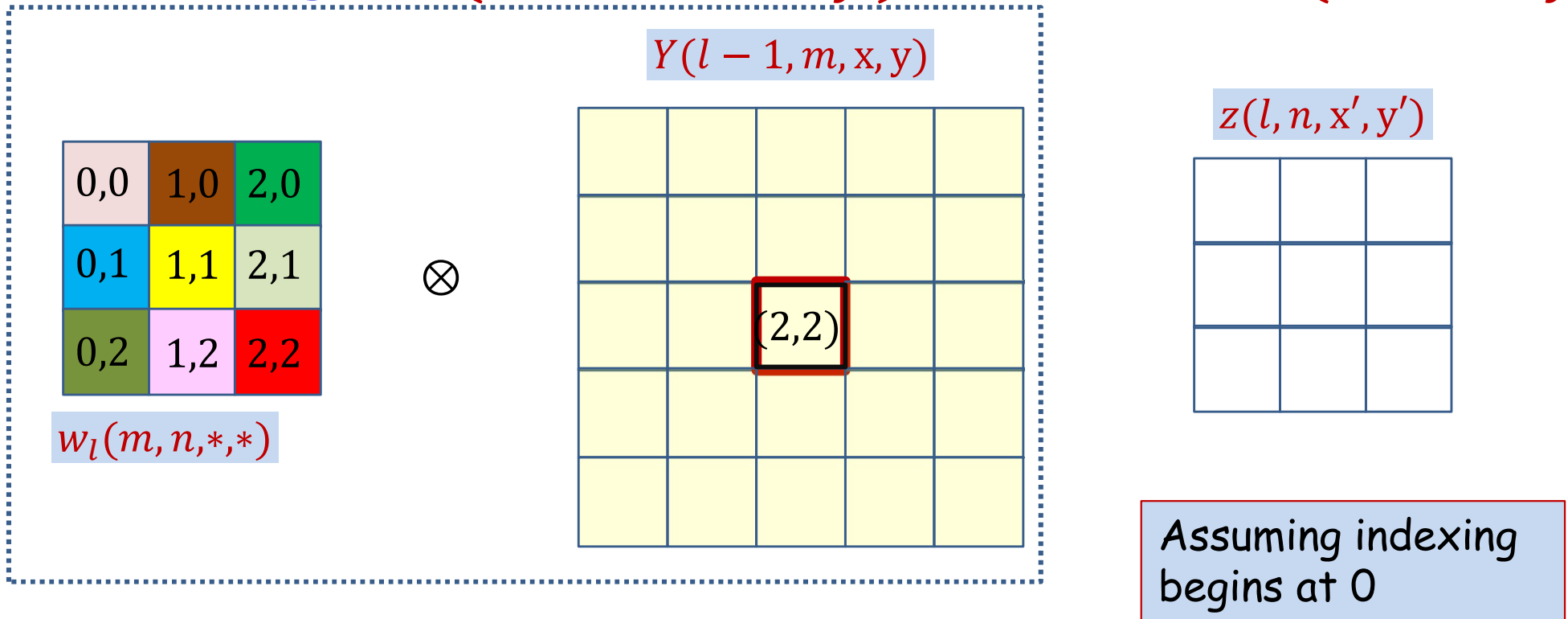


Summing over all Z maps

$$\frac{dDiv}{dY(l-1, m, x, y)} = \sum_n \sum_{x', y'} \frac{dDiv}{dz(l, n, x', y')} \frac{dz(l, n, x', y')}{dY(l-1, m, x, y)}$$

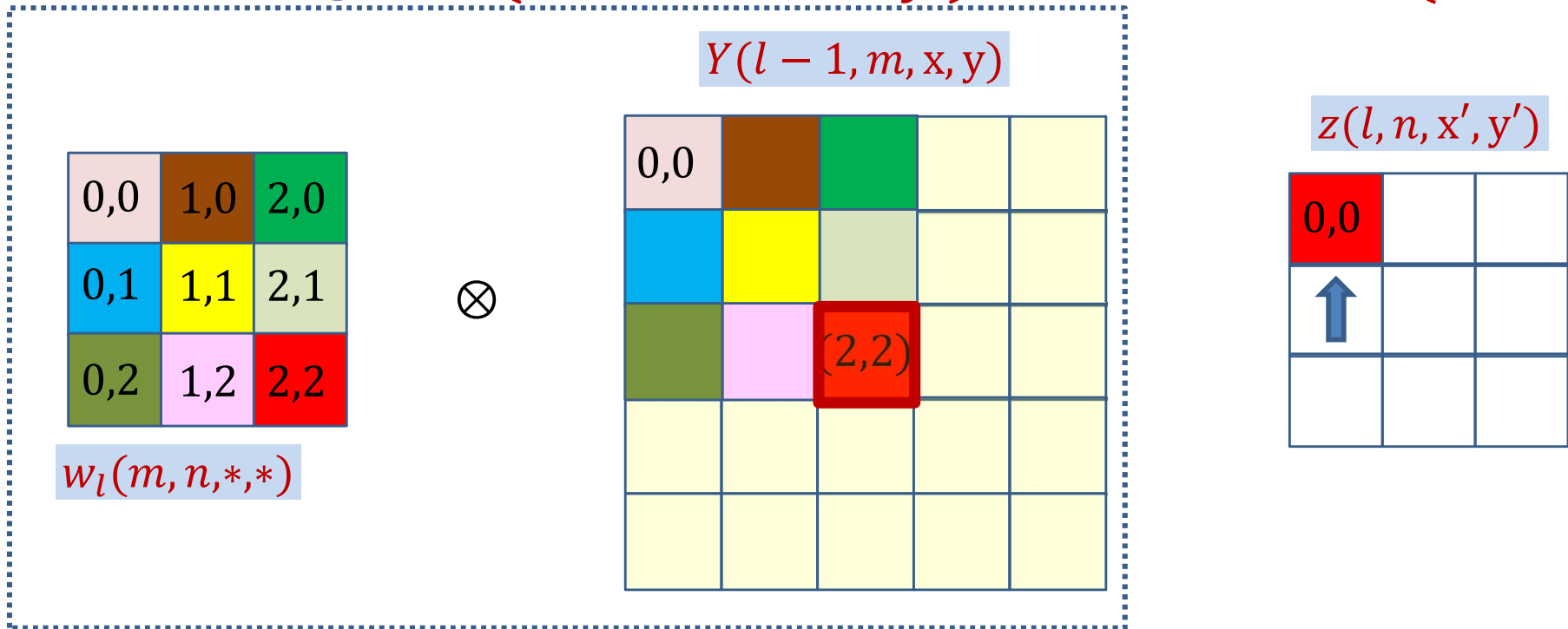
What is this?

How a single $Y(l-1, m, x, y)$ influences $z(l, n, x', y')$



- Compute how *each* x, y in Y influences various locations of z

How a single $Y(l-1, m, x, y)$ influences $z(l, n, x', y')$

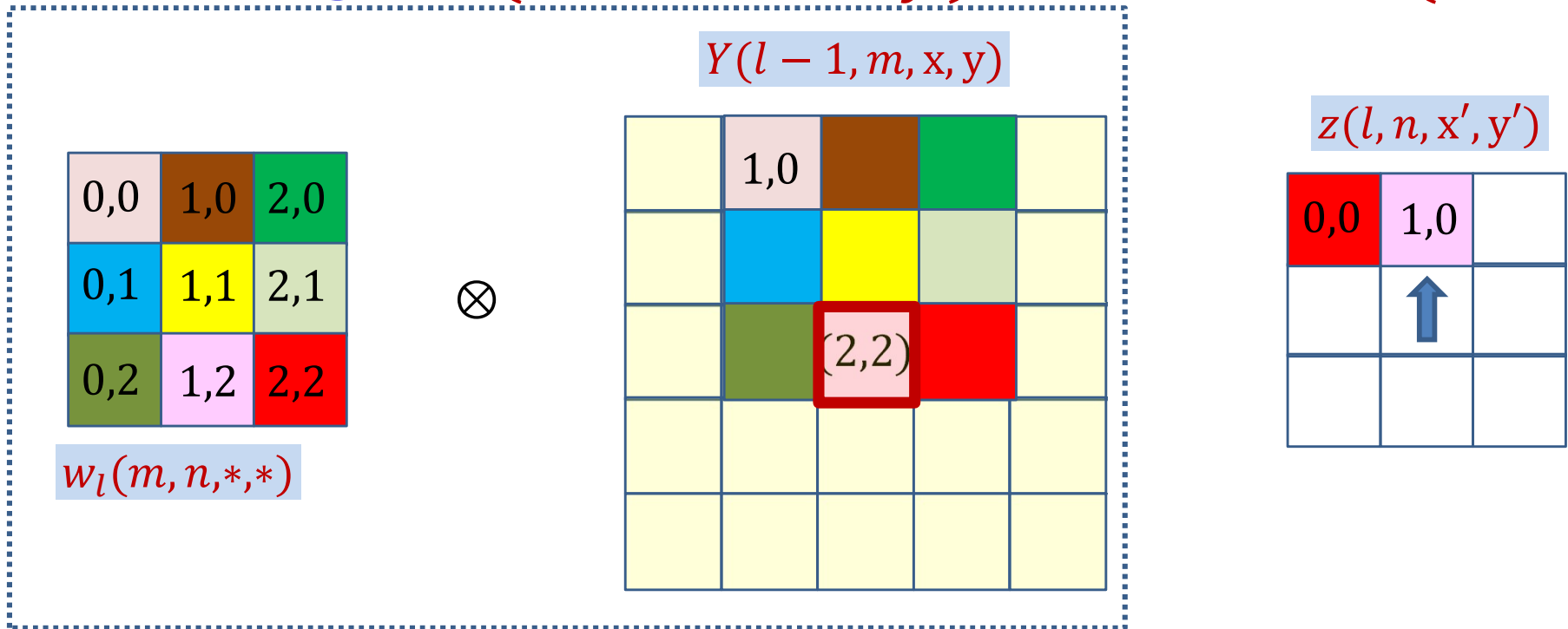


$$z(l, n, 0, 0) += Y(l-1, m, 2, 2)w_l(m, n, 2, 2)$$

- Note:** The coordinates of $z(l, n)$ and $w_l(m, n)$ sum to the coordinates of $Y(l-1, m)$

$$z(l, n, x', y') += Y(l-1, m, 2, 2)w_l(m, n, 2-x', 2-y')$$

How a single $Y(l-1, m, x, y)$ influences $z(l, n, x', y')$

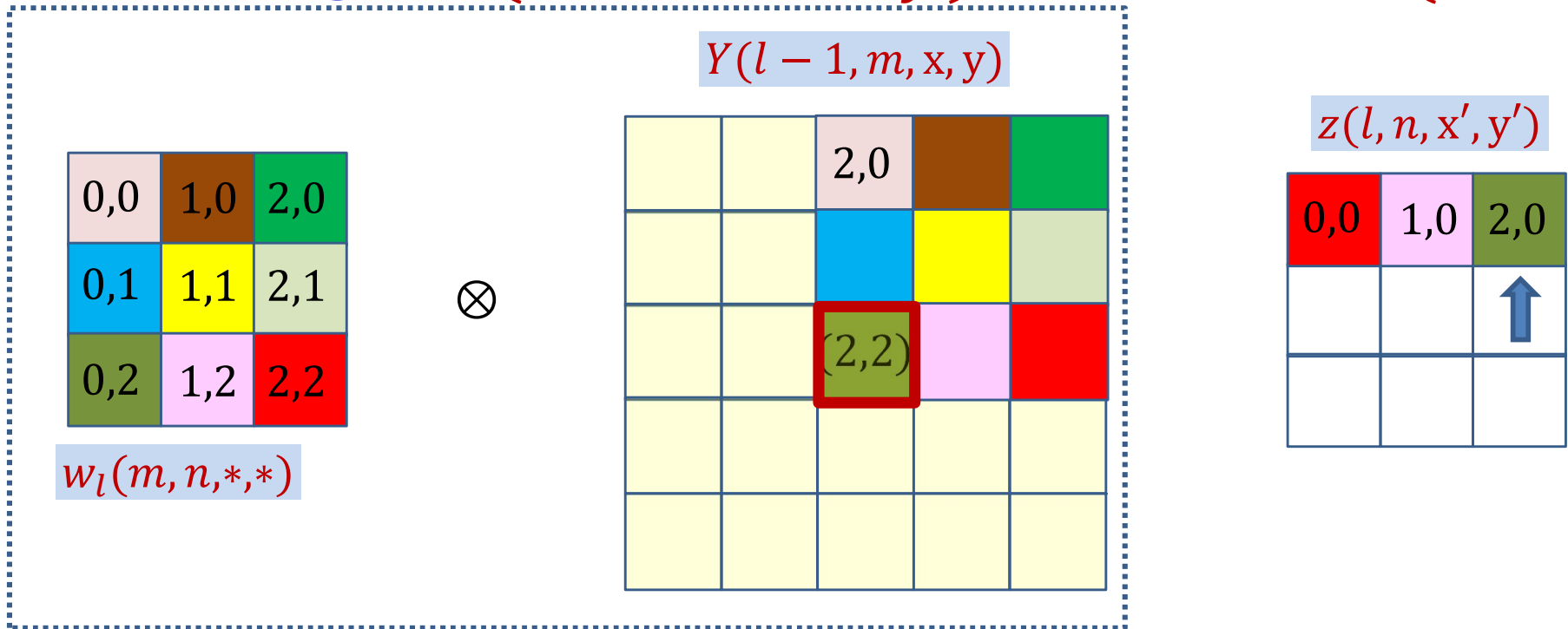


$$z(l, n, 1, 0) += Y(l-1, m, 2, 2)w_l(m, n, 1, 2)$$

- Note:** The coordinates of $z(l, n)$ and $w_l(m, n)$ sum to the coordinates of $Y(l-1, m)$

$$z(l, n, x', y') += Y(l-1, m, 2, 2)w_l(m, n, 2-x', 2-y')$$

How a single $Y(l-1, m, x, y)$ influences $z(l, n, x', y')$

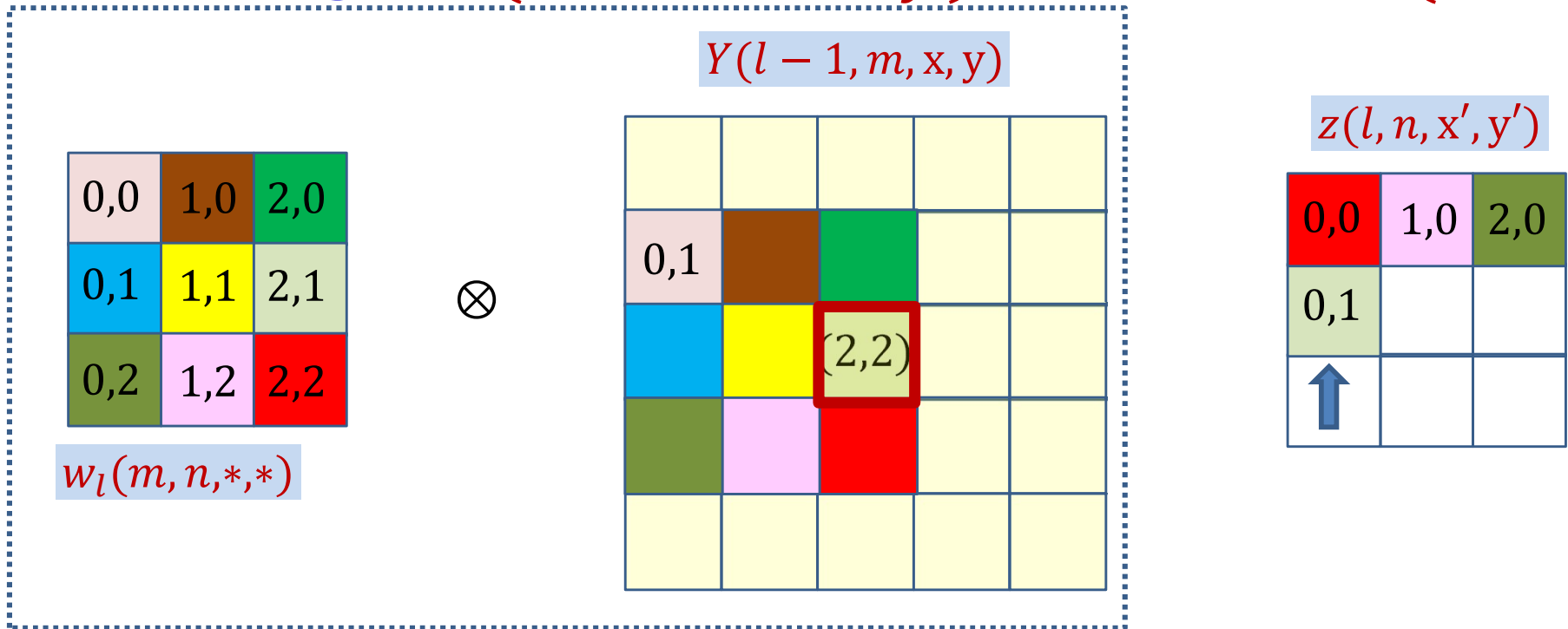


$$z(l, n, 2, 0) += Y(l-1, m, 2, 2)w_l(m, n, 0, 2)$$

- Note:** The coordinates of $z(l, n)$ and $w_l(m, n)$ sum to the coordinates of $Y(l-1, m)$

$$z(l, n, x', y') += Y(l-1, m, 2, 2)w_l(m, n, 2-x', 2-y')$$

How a single $Y(l-1, m, x, y)$ influences $z(l, n, x', y')$

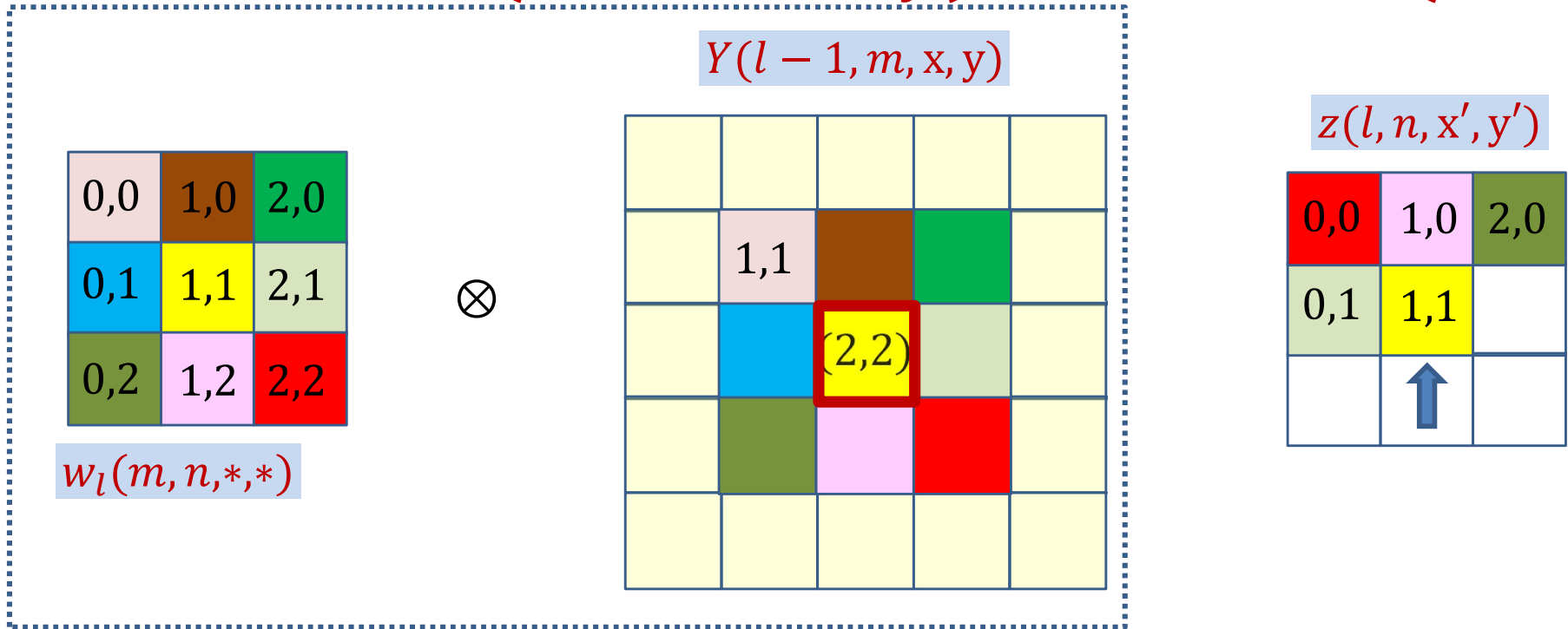


$$z(l, n, 0,1) += Y(l-1, m, 2,2)w_l(m, n, 2,1)$$

- Note:** The coordinates of $z(l, n)$ and $w_l(m, n)$ sum to the coordinates of $Y(l-1, m)$

$$z(l, n, x', y') += Y(l-1, m, 2,2)w_l(m, n, 2-x', 2-y')$$

How a single $Y(l-1, m, x, y)$ influences $z(l, n, x', y')$

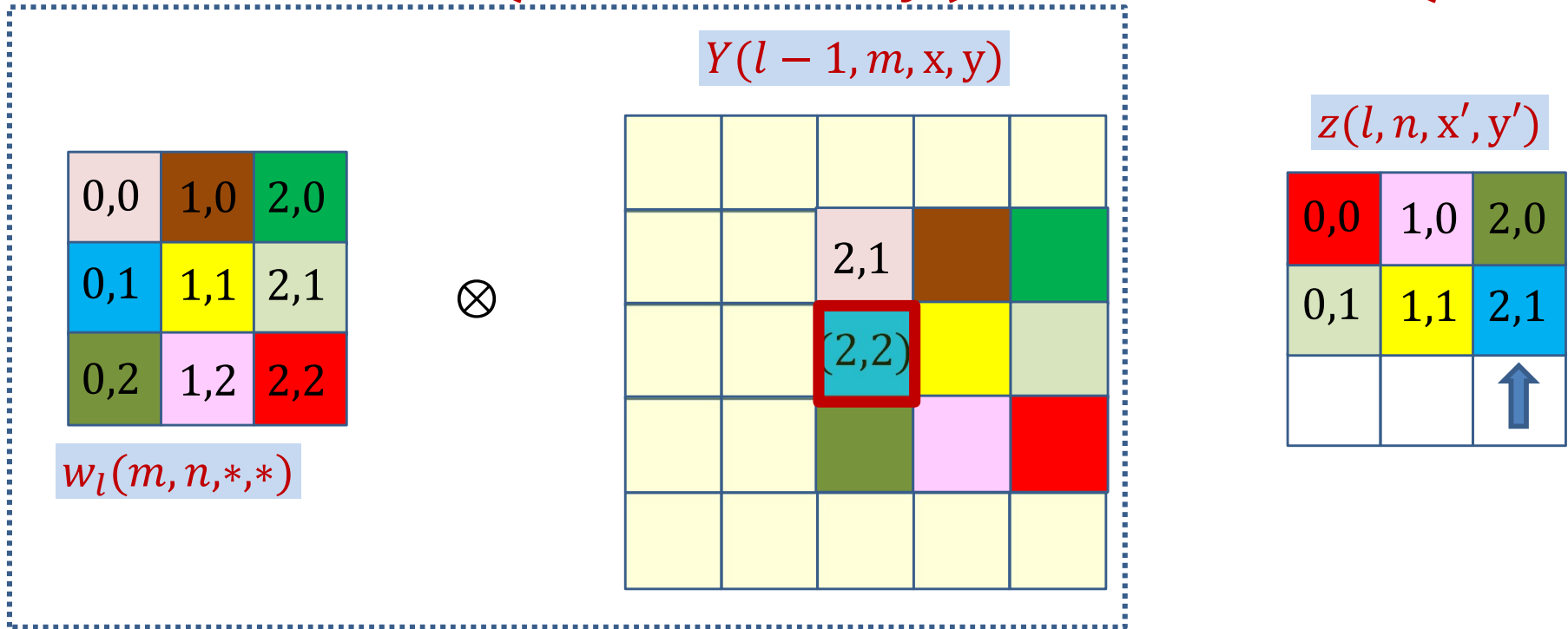


$$z(l, n, 1, 1) += Y(l-1, m, 2, 2)w_l(m, n, 1, 1)$$

- Note:** The coordinates of $z(l, n)$ and $w_l(m, n)$ sum to the coordinates of $Y(l-1, m)$

$$z(l, n, x', y') += Y(l-1, m, 2, 2)w_l(m, n, 2-x', 2-y')$$

How a single $Y(l-1, m, x, y)$ influences $z(l, n, x', y')$

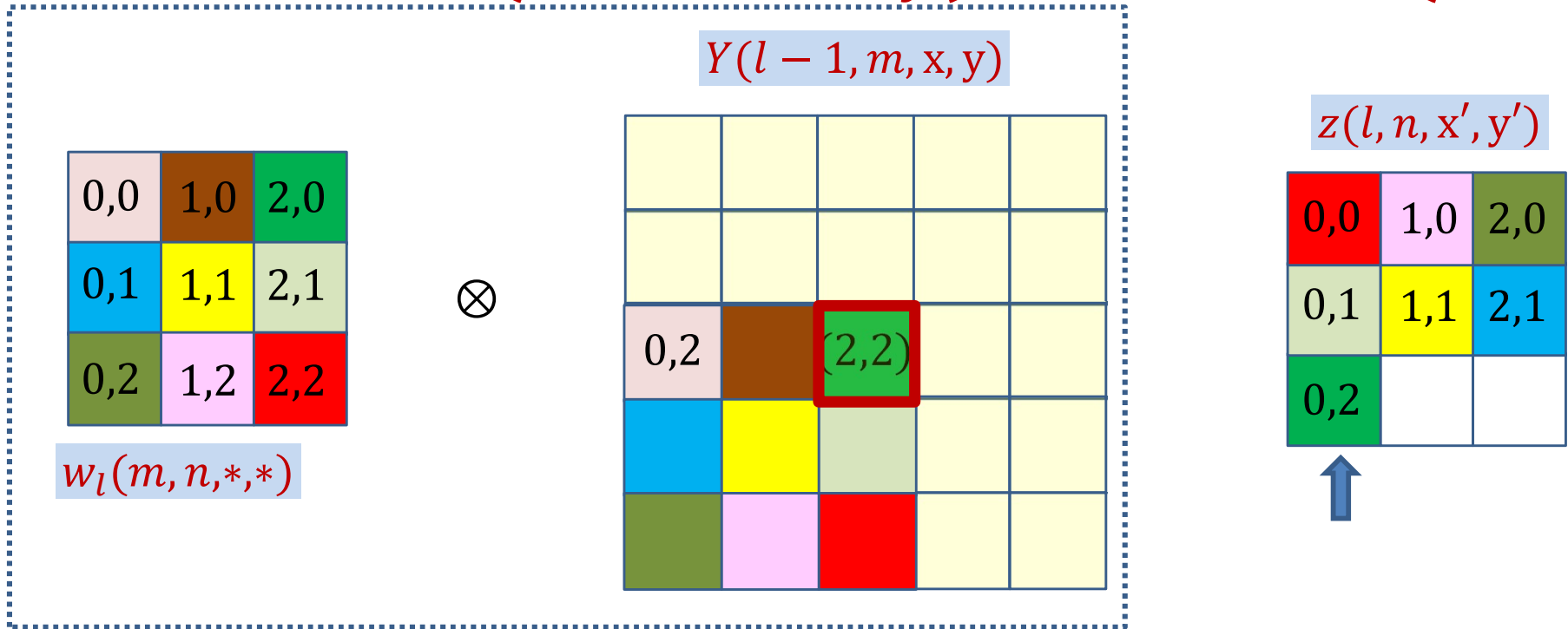


$$z(l, n, 2, 1) += Y(l-1, m, 2, 2)w_l(m, n, 0, 1)$$

- Note:** The coordinates of $z(l, n)$ and $w_l(m, n)$ sum to the coordinates of $Y(l-1, m)$

$$z(l, n, x', y') += Y(l-1, m, 2, 2)w_l(m, n, 2-x', 2-y')$$

How a single $Y(l-1, m, x, y)$ influences $z(l, n, x', y')$

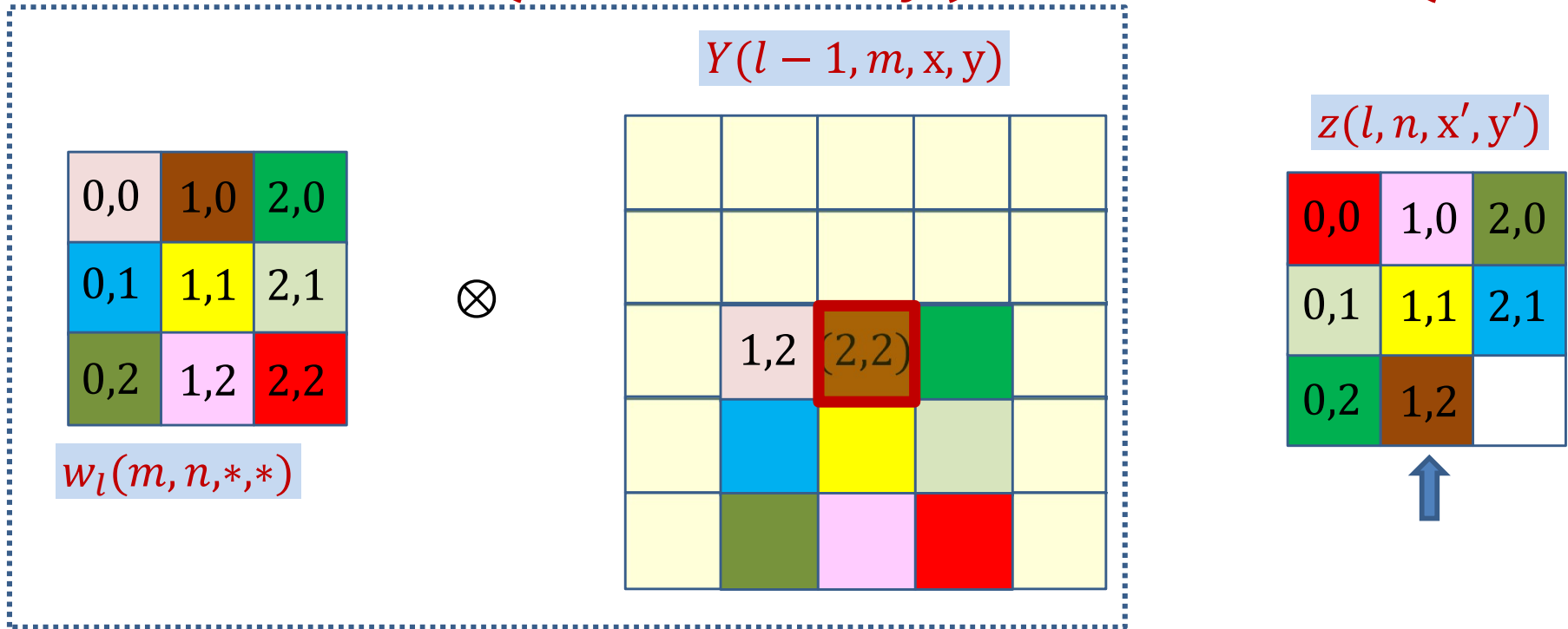


$$z(l, n, 0, 2) += Y(l-1, m, 2, 2)w_l(m, n, 2, 0)$$

- Note:** The coordinates of $z(l, n)$ and $w_l(m, n)$ sum to the coordinates of $Y(l-1, m)$

$$z(l, n, x', y') += Y(l-1, m, 2, 2)w_l(m, n, 2-x', 2-y')$$

How a single $Y(l-1, m, x, y)$ influences $z(l, n, x', y')$

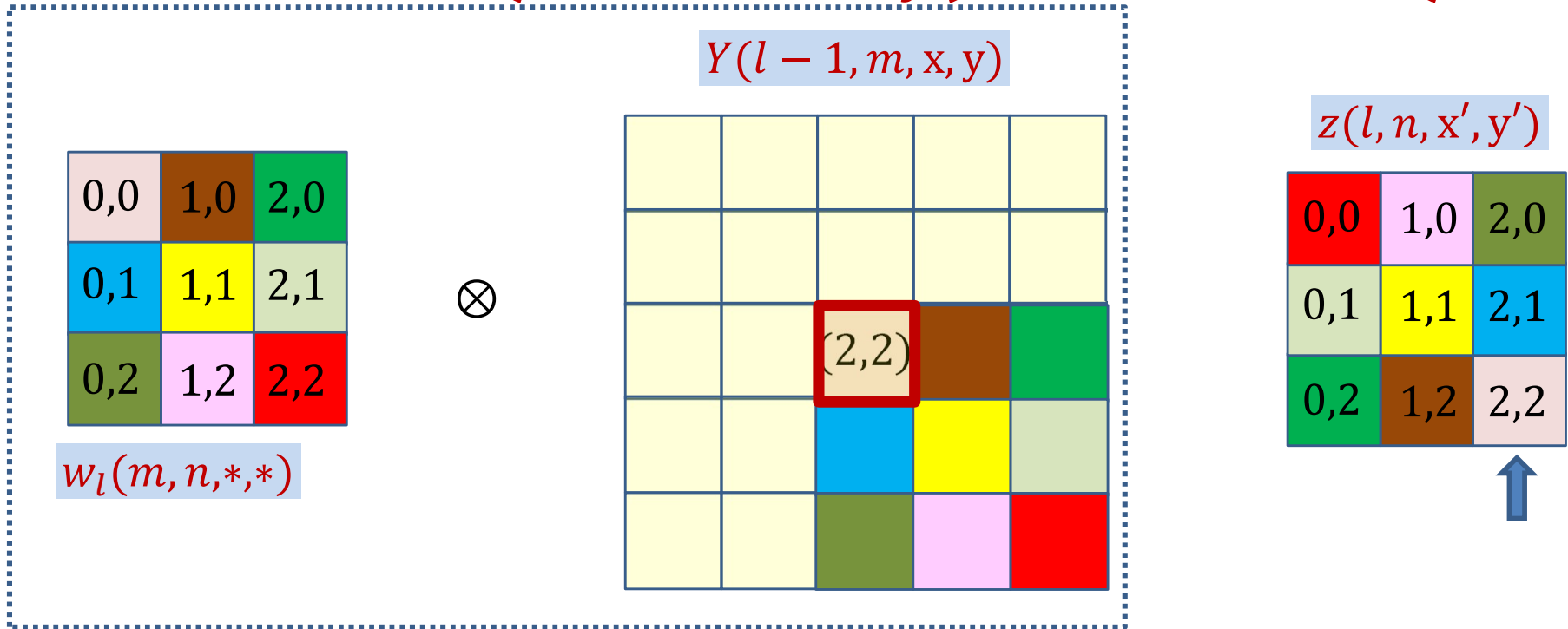


$$z(l, n, 1, 2) += Y(l-1, m, 2, 2)w_l(m, n, 2, 1)$$

- Note:** The coordinates of $z(l, n)$ and $w_l(m, n)$ sum to the coordinates of $Y(l-1, m)$

$$z(l, n, x', y') += Y(l-1, m, 2, 2)w_l(m, n, 2-x', 2-y')$$

How a single $Y(l-1, m, x, y)$ influences $z(l, n, x', y')$

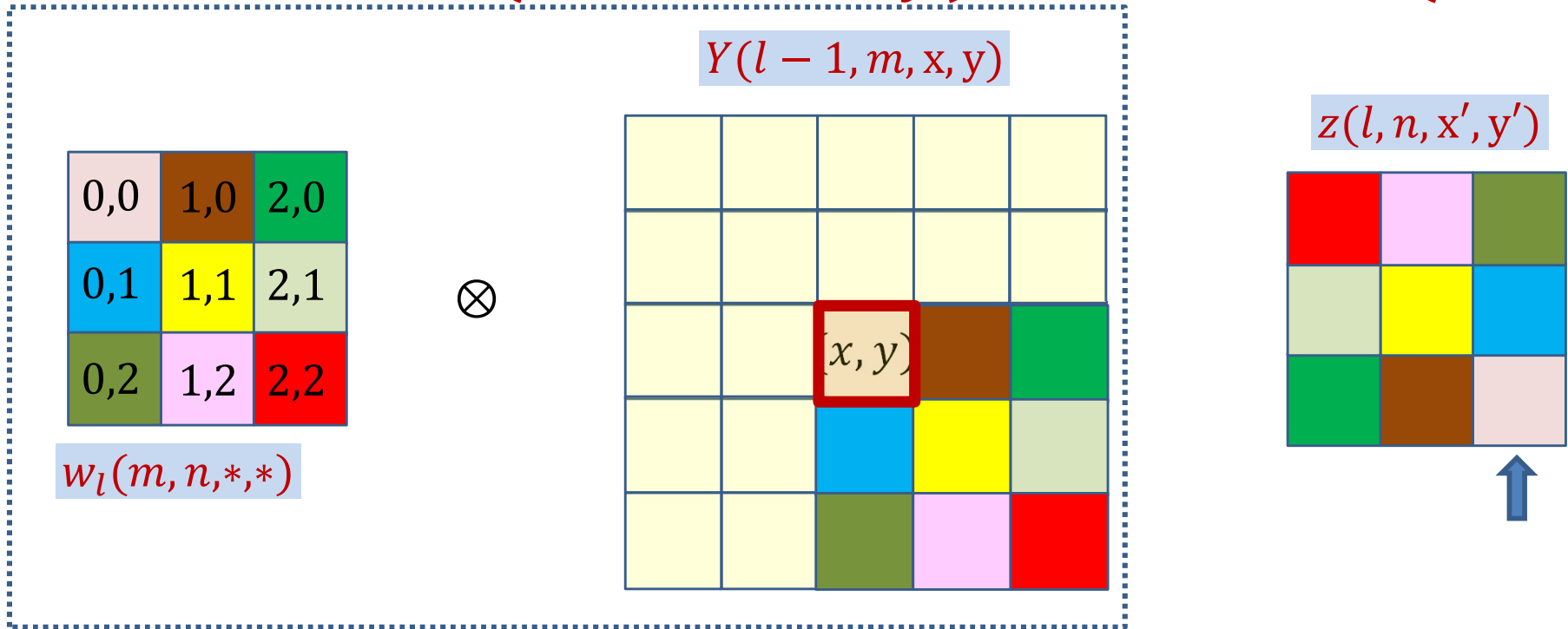


$$z(l, n, 2, 2) += Y(l-1, m, 2, 2)w_l(m, n, 0, 0)$$

- Note:** The coordinates of $z(l, n)$ and $w_l(m, n)$ sum to the coordinates of $Y(l-1, m)$

$$z(l, n, x', y') += Y(l-1, m, 2, 2)w_l(m, n, 2-x', 2-y')$$

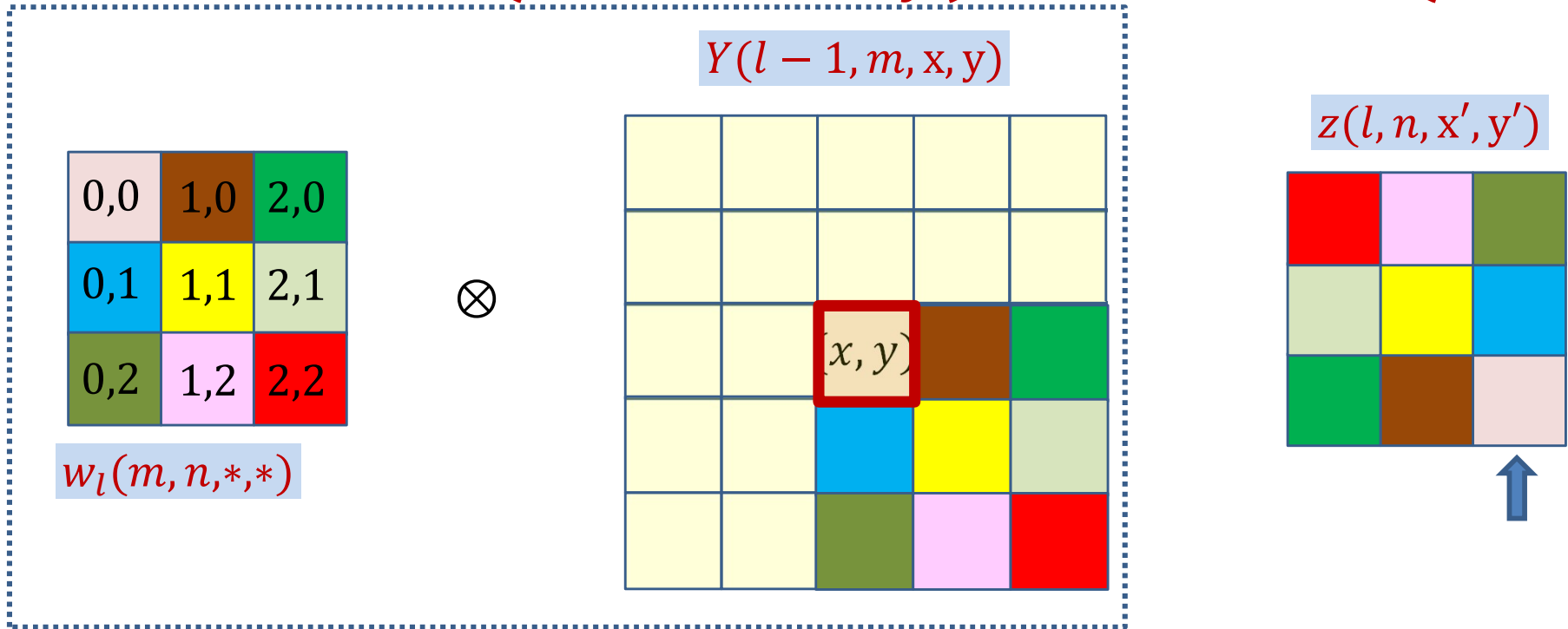
How a single $Y(l - 1, m, x, y)$ influences $z(l, n, x', y')$



$$z(l, n, x', y') += Y(l - 1, m, x, y) w_l(m, n, x - x', y - y')$$

- **Note:** The coordinates of $z(l, n)$ and $w_l(m, n)$ sum to the coordinates of $Y(l - 1, m)$

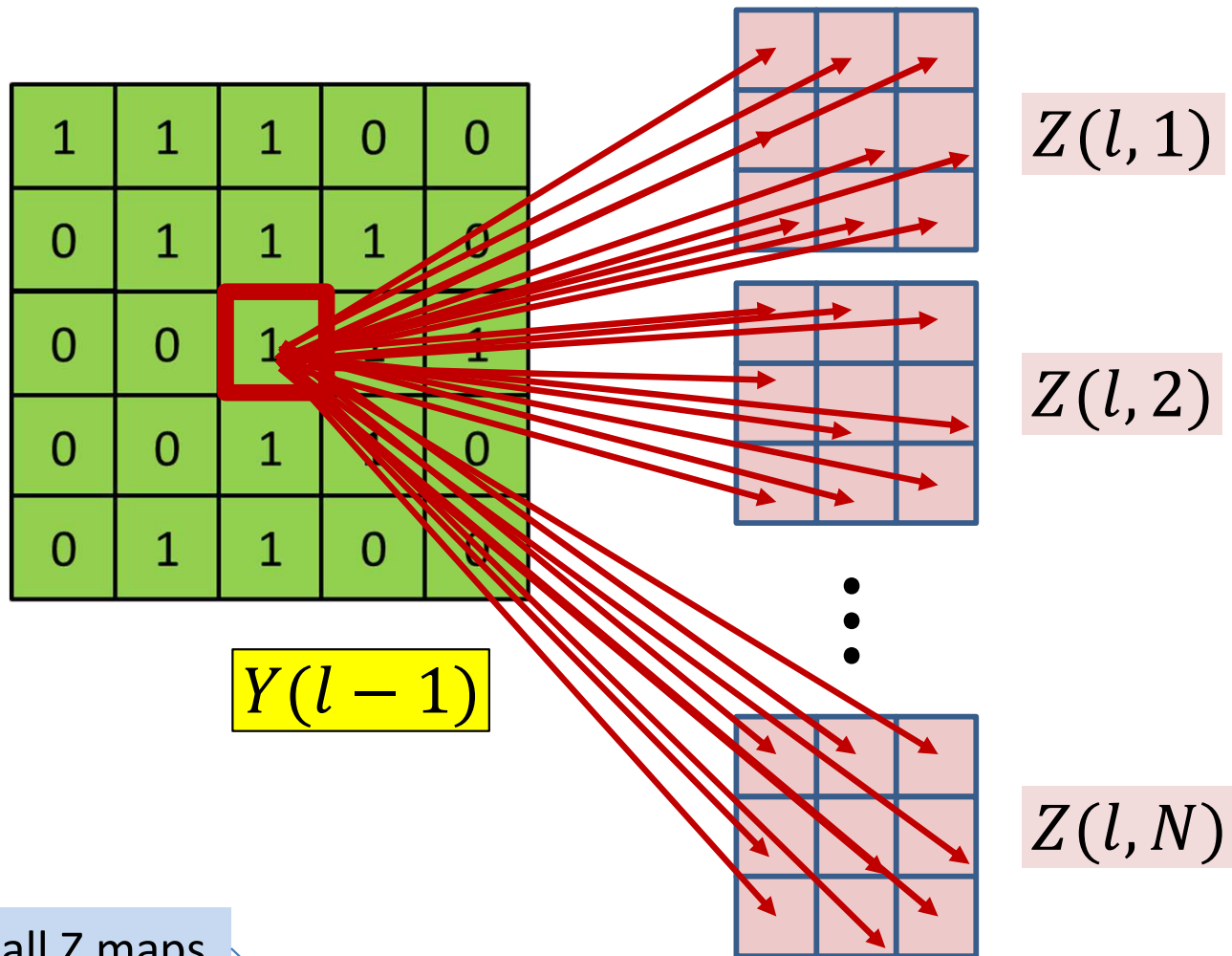
How a single $Y(l-1, m, x, y)$ influences $z(l, n, x', y')$



$$z(l, n, x', y') += Y(l-1, m, x, y) w_l(m, n, x-x', y-y')$$

$$\frac{dz(l, n, x', y')}{dY(l-1, m, x, y)} = w_l(m, n, x-x', y-y')$$

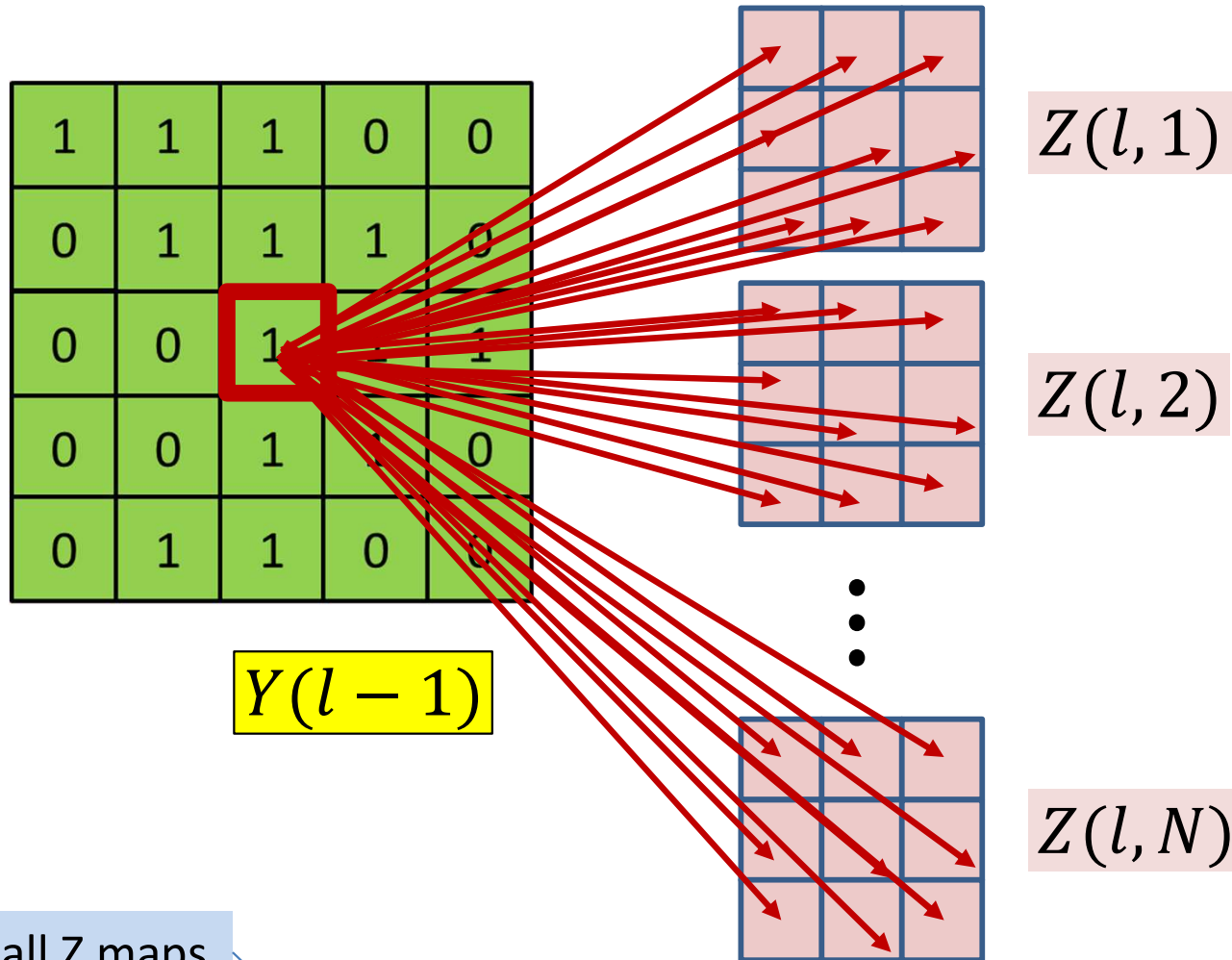
BP: Convolutional layer



Summing over all Z maps

$$\frac{dDiv}{dY(l-1, m, x, y)} = \sum_n \sum_{x', y'} \frac{dDiv}{dz(l, n, x', y')} \frac{dz(l, n, x', y')}{dY(l-1, m, x, y)}$$

BP: Convolutional layer



Summing over all Z maps

$$\frac{dDiv}{dY(l-1, m, x, y)} = \sum_n \sum_{x', y'} \frac{dDiv}{dz(l, n, x', y')} w_l(m, n, x - x', y - y')$$

Poll 2: @758, @759

In order to compute the derivative at a single affine element $Y(l,m,x,y)$, we must consider the contributions of *every* position of *every* affine map at the next layer: True or false

- True
- False

The derivative for a single affine element $Y(l,m,x,y)$ will require summing over every position of every Z map in the next layer: True or false

- True
- False

Poll 2

In order to compute the derivative at a single affine element $Y(l,m,x,y)$, we must consider the contributions of *every* position of *every* affine map at the next layer: True or false

- True
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The derivative for an single affine element $Y(l,m,x,y)$ will require summing over every position of every Z map in the next layer: True or false

- True
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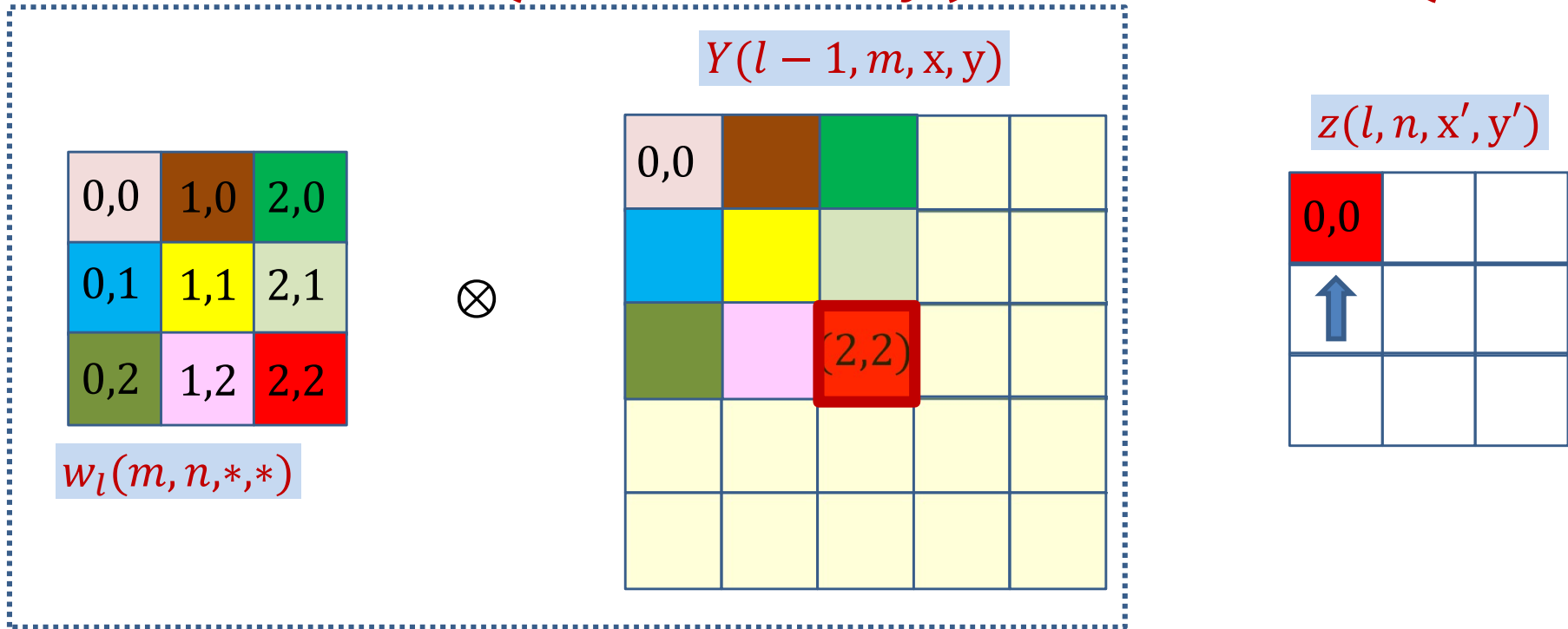
Computing derivative for $Y(l - 1, \mathbf{m}, *, *)$

- The derivatives for every element of every map in $Y(l - 1)$ by direct implementation of the formula:

$$\frac{dDiv}{dY(l - 1, m, x, y)} = \sum_n \sum_{x', y'} \frac{dDiv}{dz(l, n, x', y')} w_l(m, n, x - x', y - y')$$

- But this is actually a convolution!
 - Let's see how

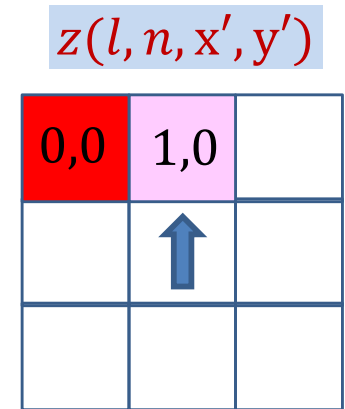
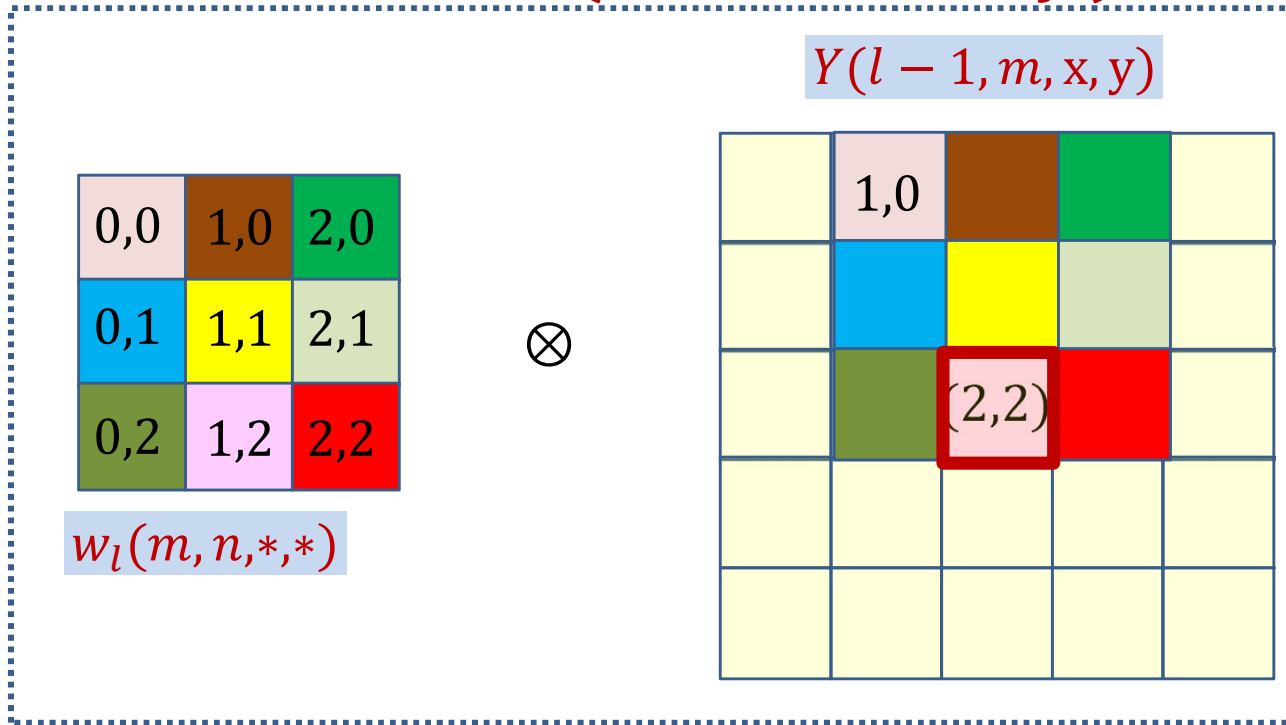
How a single $Y(l-1, m, x, y)$ influences $z(l, n, x', y')$



$$z(l, n, 0, 0) += Y(l-1, m, 2, 2)w_l(m, n, 2, 2)$$

$$\frac{dDiv}{dY(l-1, m, 2, 2)} += \frac{dDiv}{dz(l, n, 0, 0)} w_l(m, n, 2, 2)$$

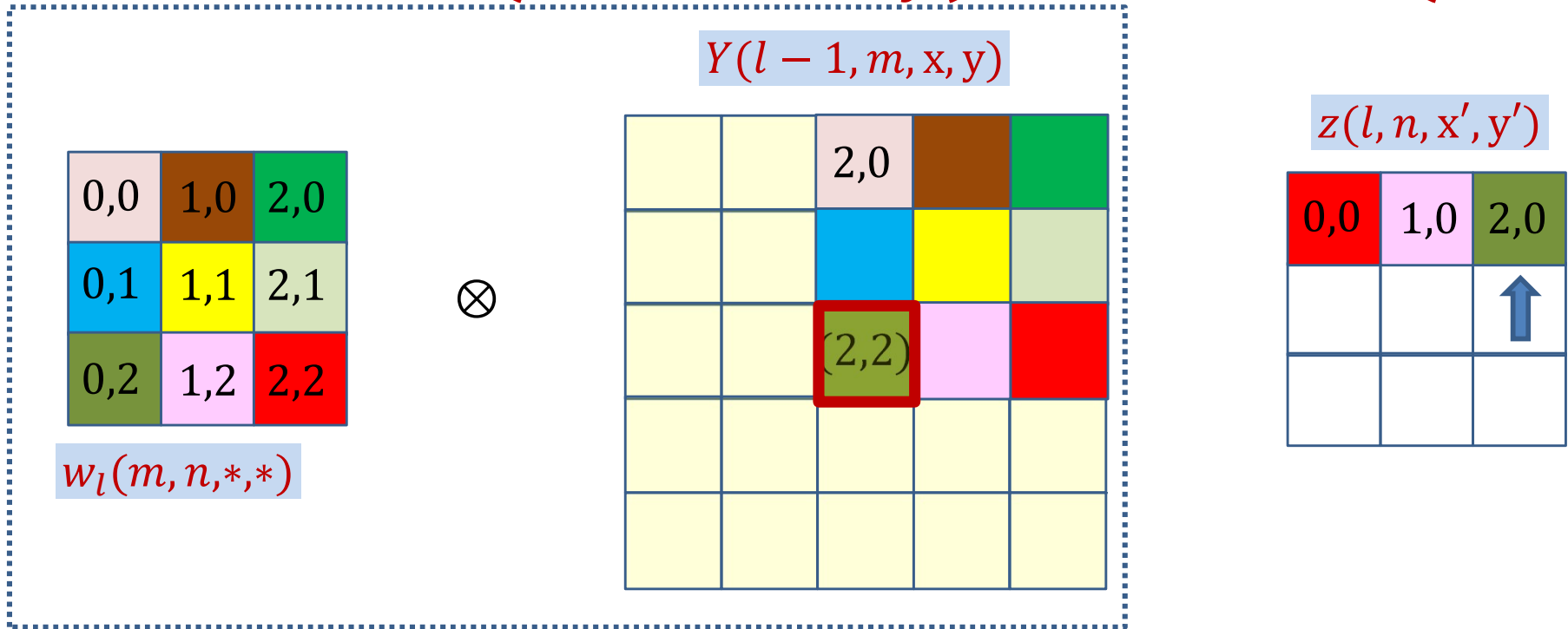
How a single $Y(l-1, m, x, y)$ influences $z(l, n, x', y')$



$$z(l, n, 1,0) += Y(l-1, m, 2,2)w_l(m, n, 1,2)$$

$$\frac{dDiv}{dY(l-1, m, 2,2)} += \frac{dDiv}{dz(l, n, 1,0)} w_l(m, n, 1,2)$$

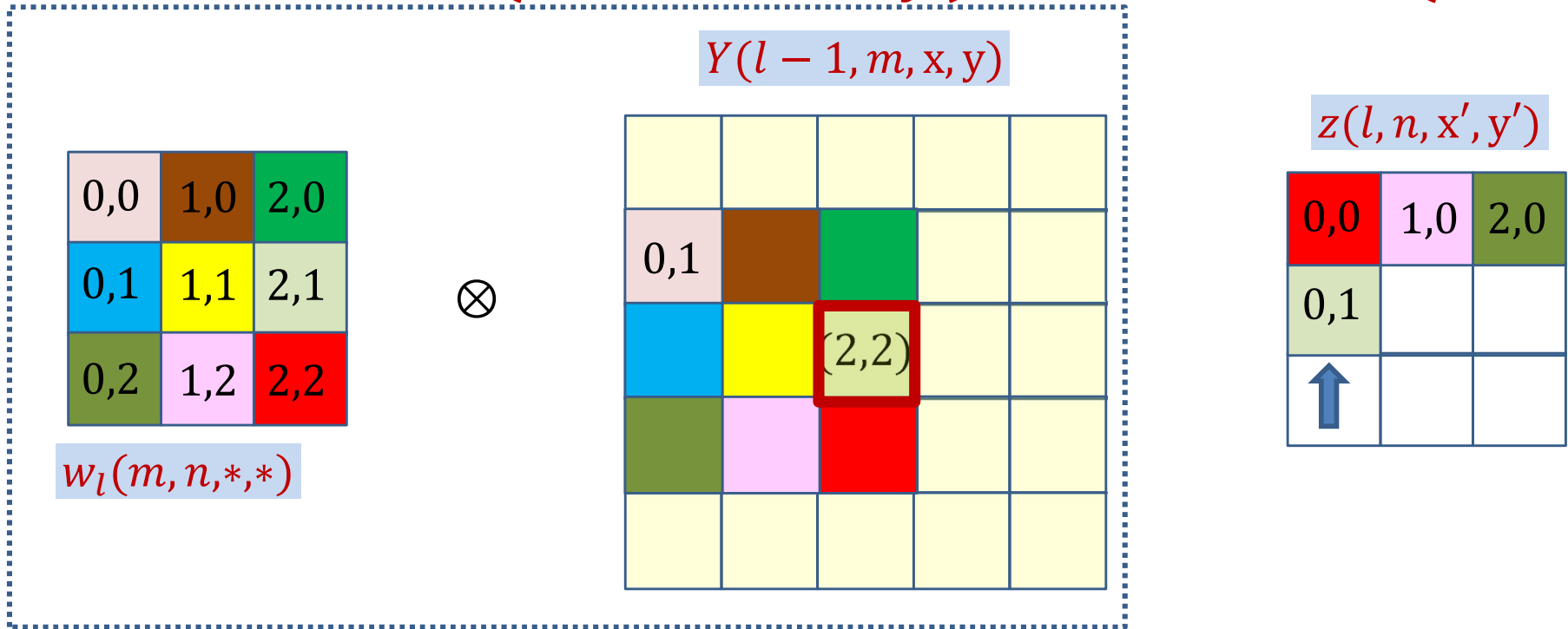
How a single $Y(l-1, m, x, y)$ influences $z(l, n, x', y')$



$$z(l, n, 2,0) += Y(l-1, m, 2,2)w_l(m, n, 0,2)$$

$$\frac{dDiv}{dY(l-1, m, 2,2)} += \frac{dDiv}{dz(l, n, 2,0)} w_l(m, n, 0,2)$$

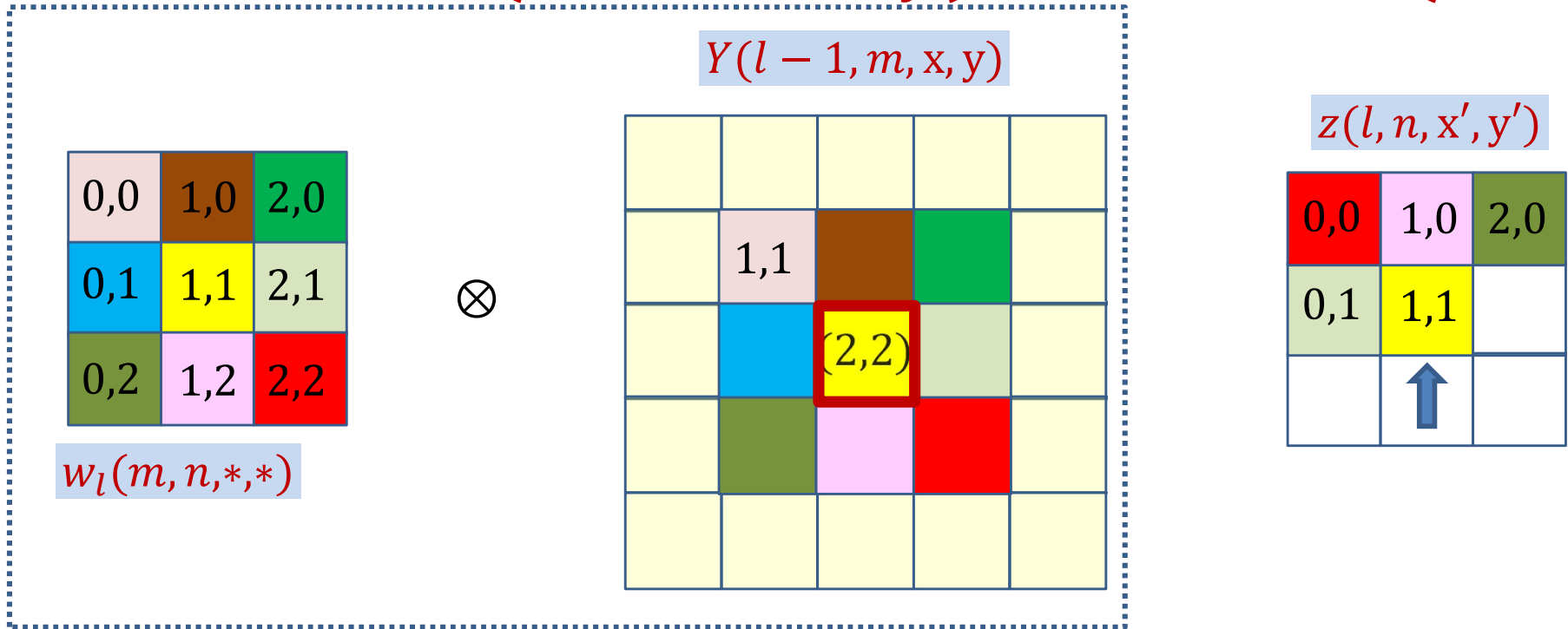
How a single $Y(l-1, m, x, y)$ influences $z(l, n, x', y')$



$$z(l, n, 0,1) += Y(l-1, m, 2,2)w_l(m, n, 2,1)$$

$$\frac{dDiv}{dY(l-1, m, 2,2)} += \frac{dDiv}{dz(l, n, 0,1)} w_l(m, n, 2,1)$$

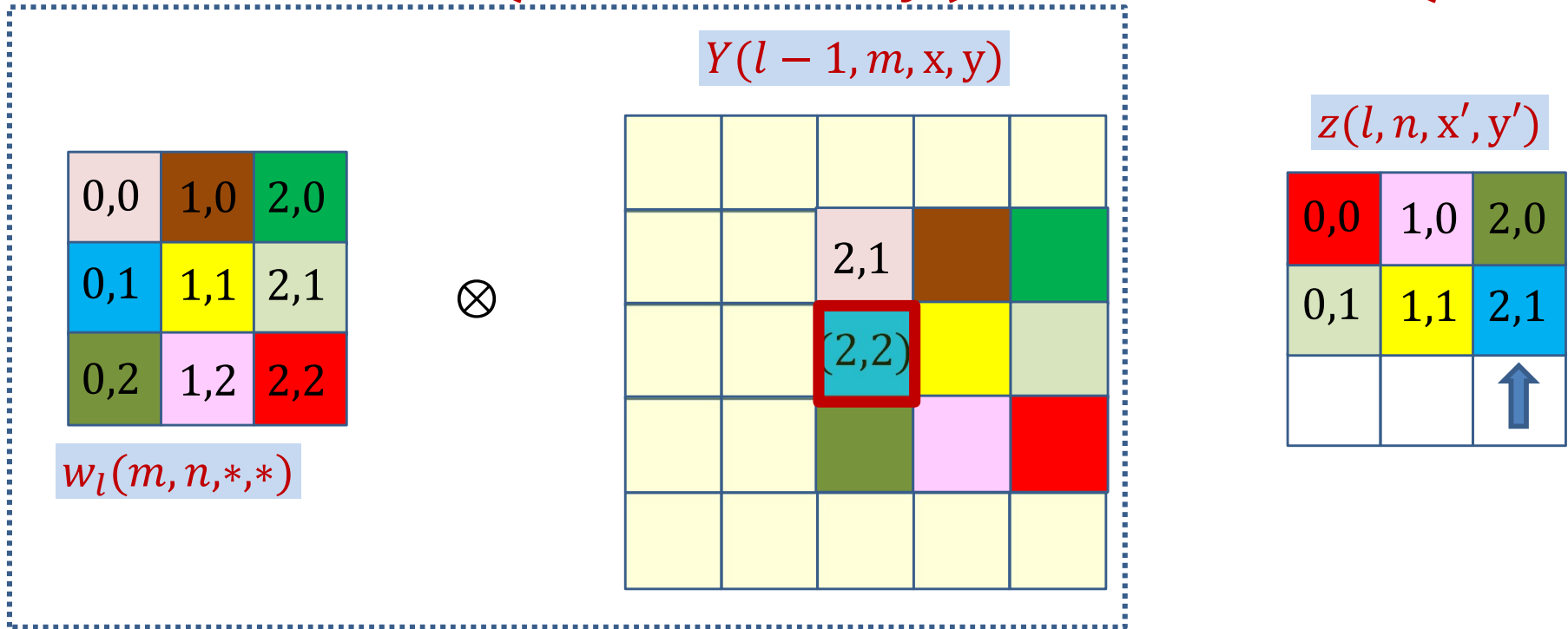
How a single $Y(l-1, m, x, y)$ influences $z(l, n, x', y')$



$$z(l, n, 1, 1) += Y(l-1, m, 2, 2) w_l(m, n, 1, 1)$$

$$\frac{dDiv}{dY(l-1, m, 2, 2)} += \frac{dDiv}{dz(l, n, 1, 1)} w_l(m, n, 1, 1)$$

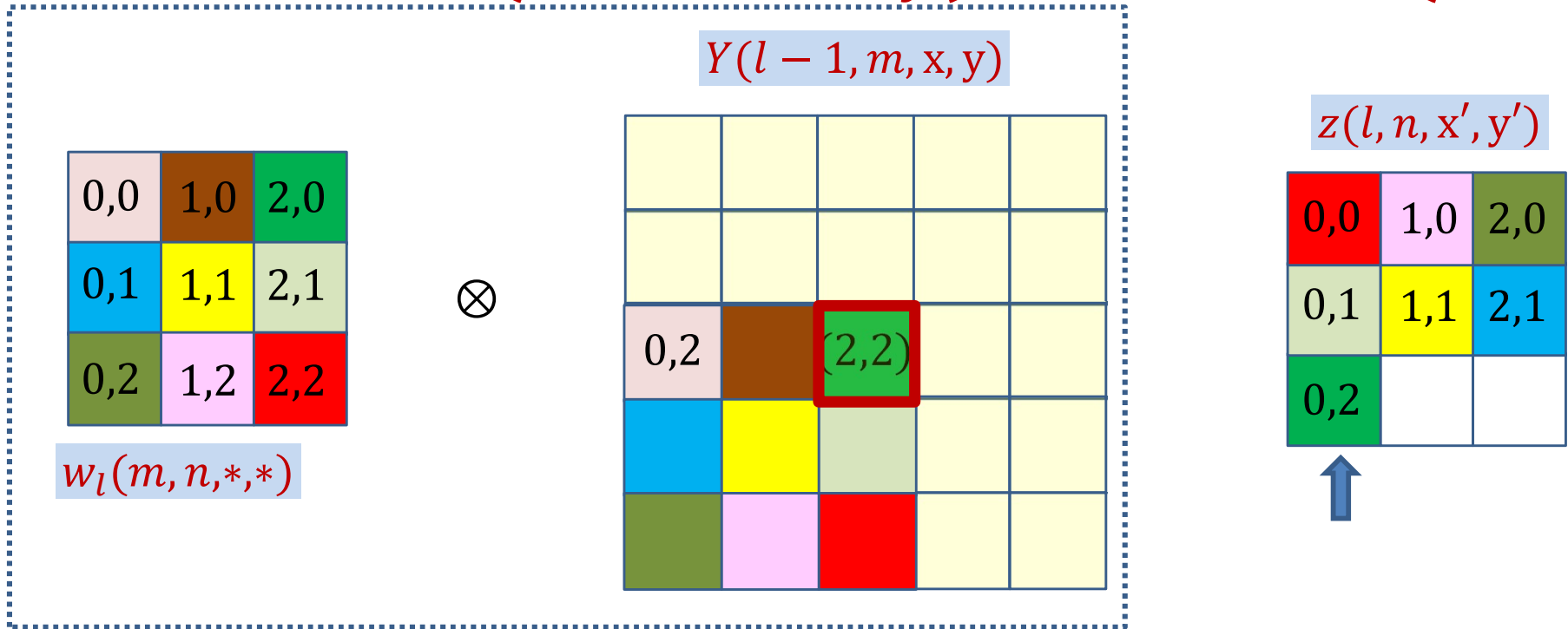
How a single $Y(l-1, m, x, y)$ influences $z(l, n, x', y')$



$$z(l, n, 2, 1) += Y(l-1, m, 2, 2) w_l(m, n, 0, 1)$$

$$\frac{dDiv}{dY(l-1, m, 2, 2)} += \frac{dDiv}{dz(l, n, 2, 1)} w_l(m, n, 0, 1)$$

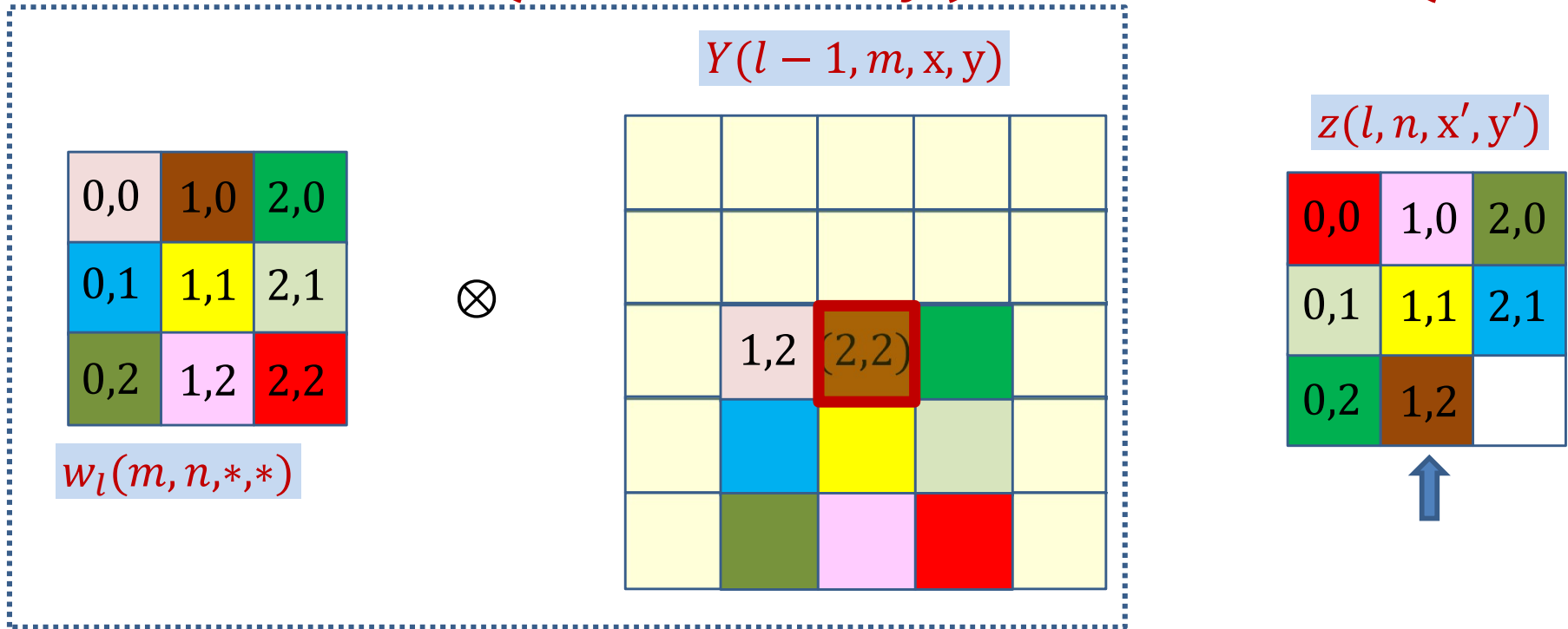
How a single $Y(l-1, m, x, y)$ influences $z(l, n, x', y')$



$$z(l, n, 0, 2) += Y(l-1, m, 2, 2)w_l(m, n, 2, 0)$$

$$\frac{dDiv}{dY(l-1, m, 2, 2)} += \frac{dDiv}{dz(l, n, 0, 2)} w_l(m, n, 2, 0)$$

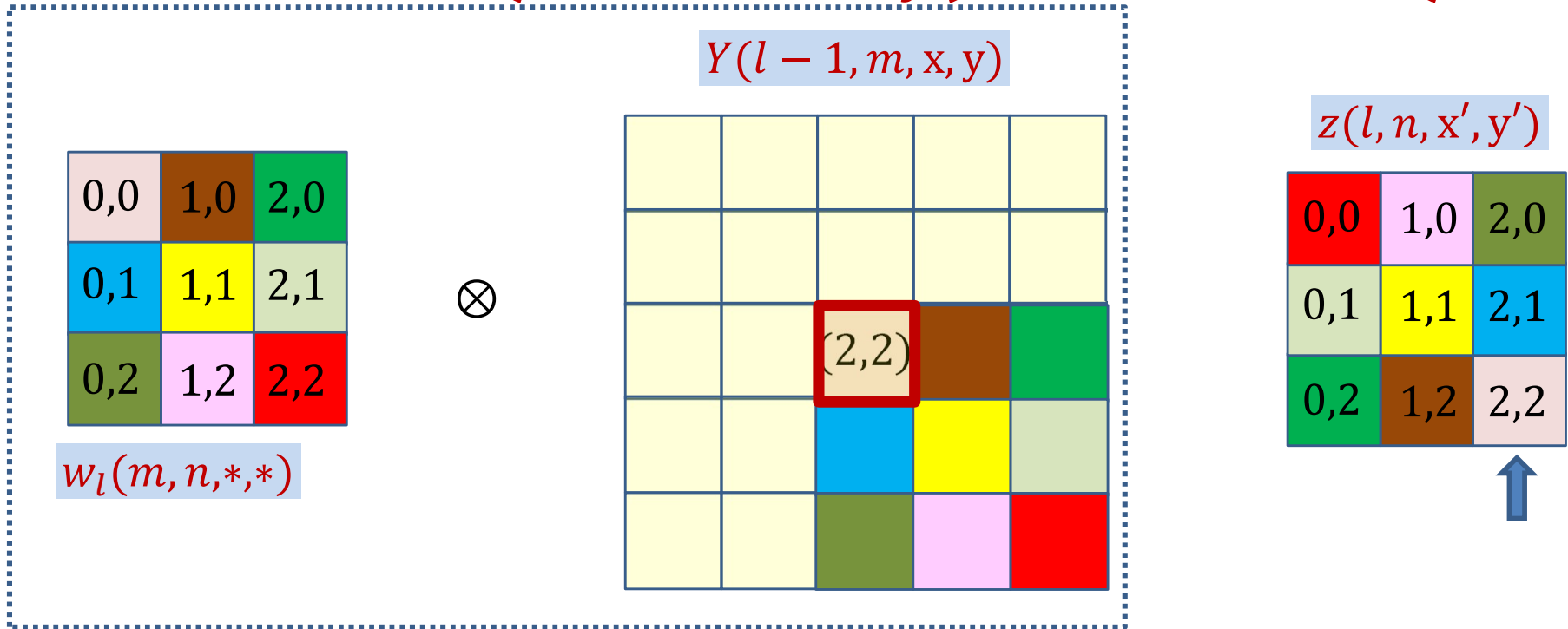
How a single $Y(l-1, m, x, y)$ influences $z(l, n, x', y')$



$$z(l, n, 1, 2) += Y(l-1, m, 2, 2) w_l(m, n, 2, 1)$$

$$\frac{dDiv}{dY(l-1, m, 2, 2)} += \frac{dDiv}{dz(l, n, 1, 2)} w_l(m, n, 1, 0)$$

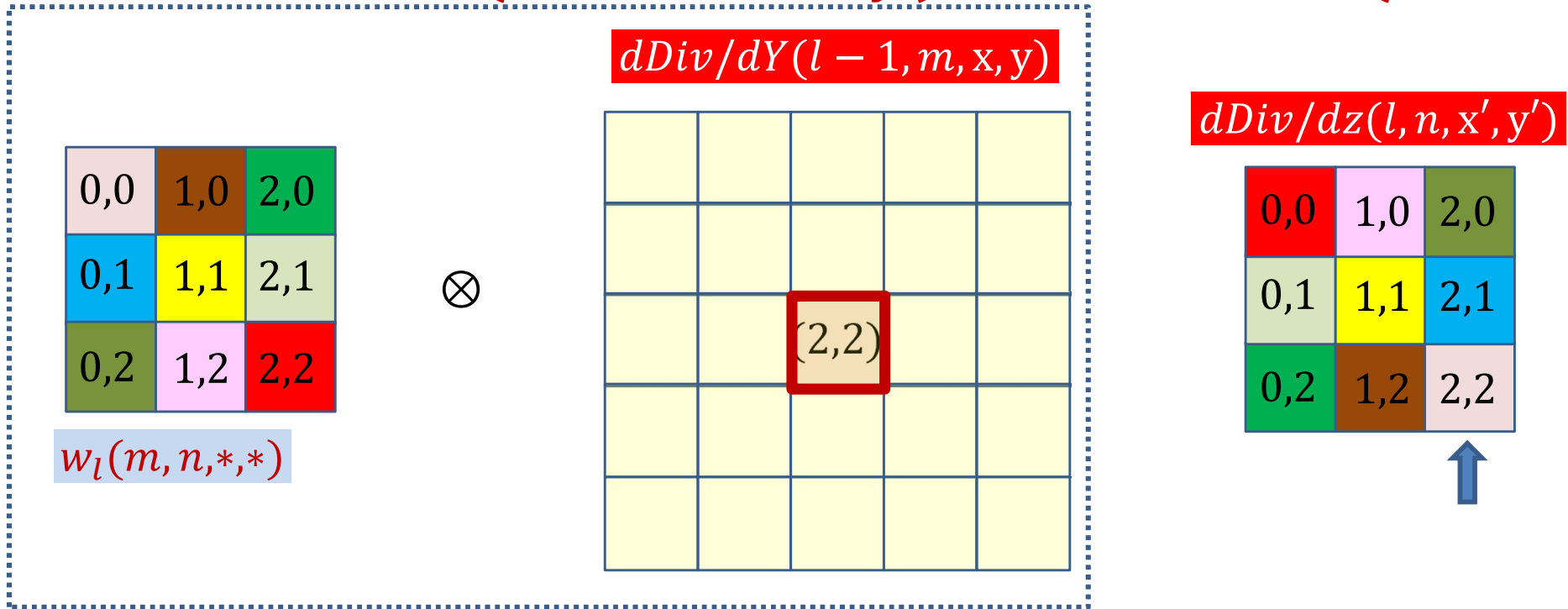
How a single $Y(l-1, m, x, y)$ influences $z(l, n, x', y')$



$$z(l, n, 2, 2) += Y(l-1, m, 2, 2) w_l(m, n, 0, 0)$$

$$\frac{dDiv}{dY(l-1, m, 2, 2)} += \frac{dDiv}{dz(l, n, 2, 2)} w_l(m, n, 0, 0)$$

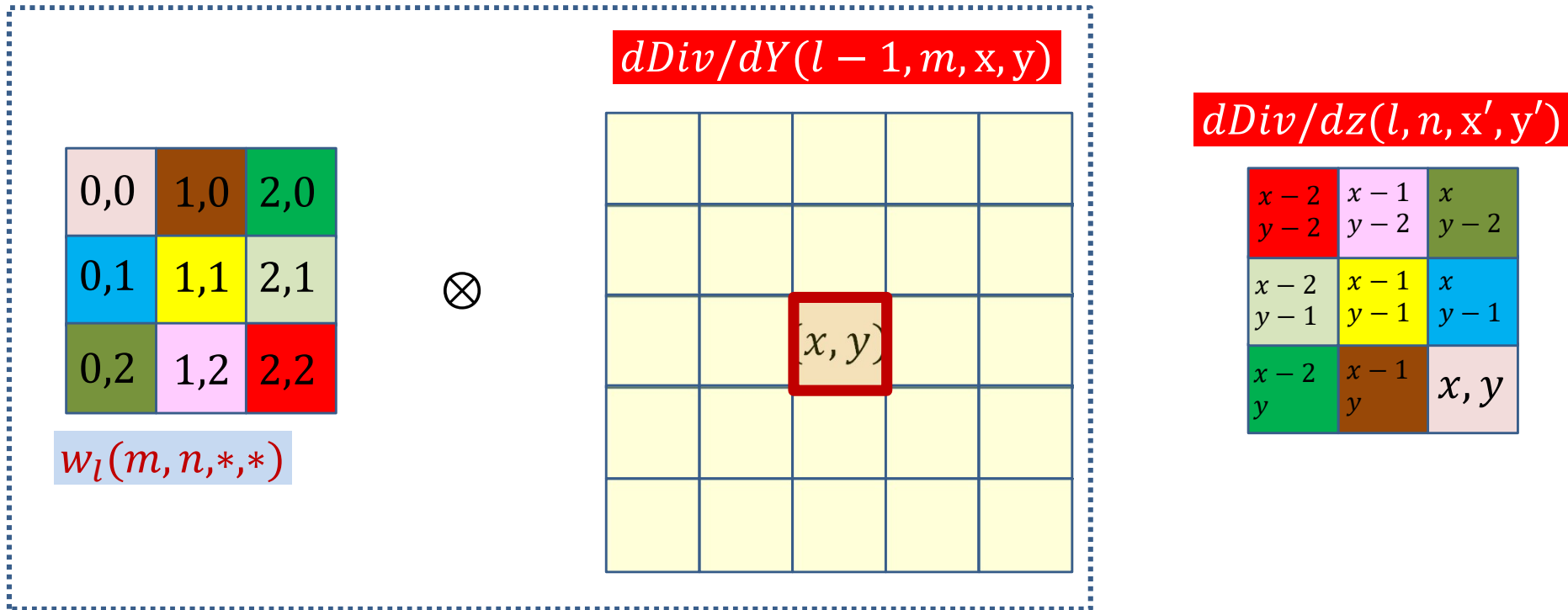
How a single $Y(l-1, m, x, y)$ influences $z(l, n, x', y')$



$$\frac{dDiv}{dY(l-1, m, 2, 2)} = \sum_{x', y'} \frac{dDiv}{dz(l, n, x', y')} w_l(m, n, 2 - x', 2 - y')$$

- The derivative at $Y(l-1, m, 2, 2)$ is the sum of component-wise product of the filter elements (shown by color) and the elements of the derivative at $z(l, m, \dots)$

Derivative at $Y(l-1, m, x, y)$ from a single $Z(l, n)$ map



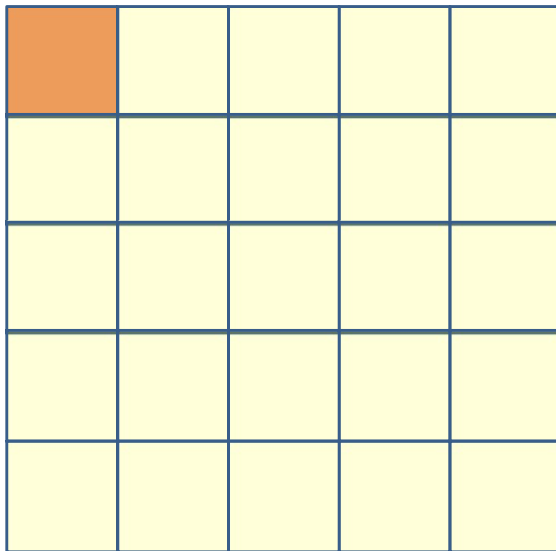
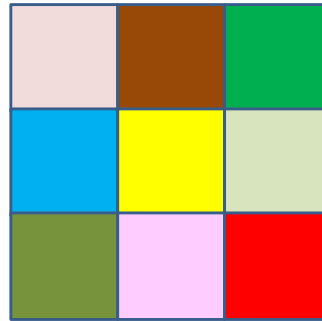
$$z(l, n, x', y') += Y(l-1, m, x, y) w_l(m, n, x-x', y-y')$$

$$\frac{dDiv}{dY(l-1, m, x, y)} += \sum_{x', y'} \frac{dDiv}{dz(l, n, x', y')} w_l(m, n, x-x', y-y')$$

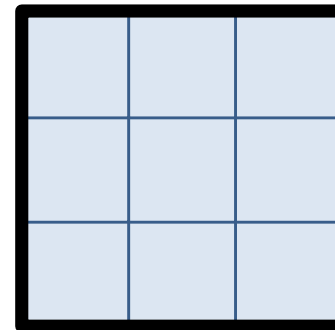
Contribution of the entire n th affine map $z(l, n, *, *)$

Derivative at $Y(l-1, m)$ from a single $Z(l, n)$ map

$w_l(m, n, *, *)$



=

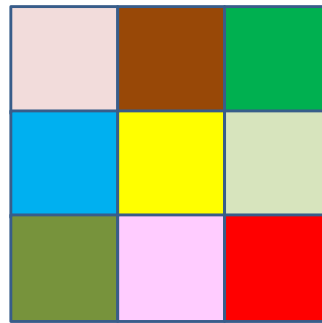


$$\frac{\partial Div}{\partial y(l-1, m, x, y)}$$

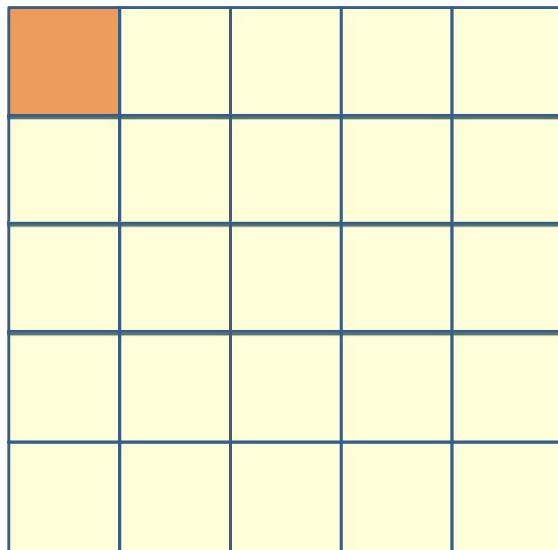
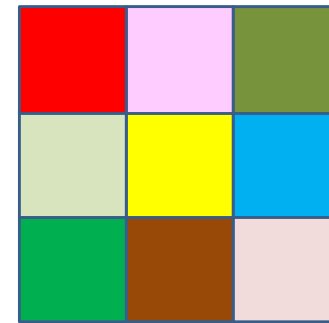
$$\frac{\partial Div}{\partial z(l, n, x', y')}$$

Derivative at $Y(l-1, m)$ from a single $Z(l, n)$ map

$w_l(m, n, *, *)$

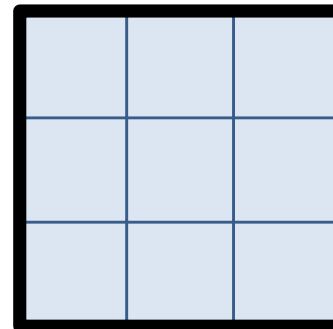


flip
→



=

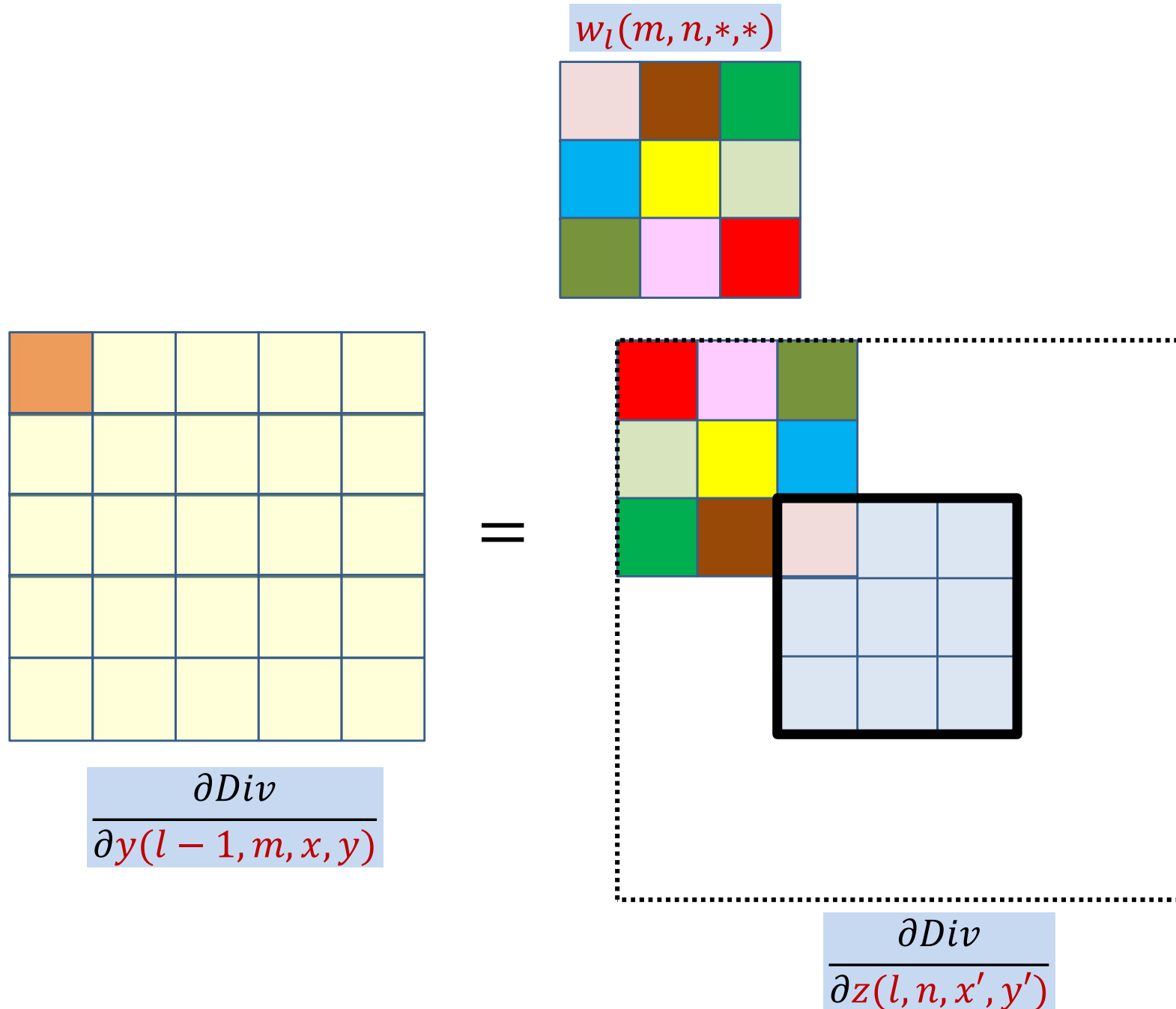
Zero pad with K-1 rows
and cols on every side



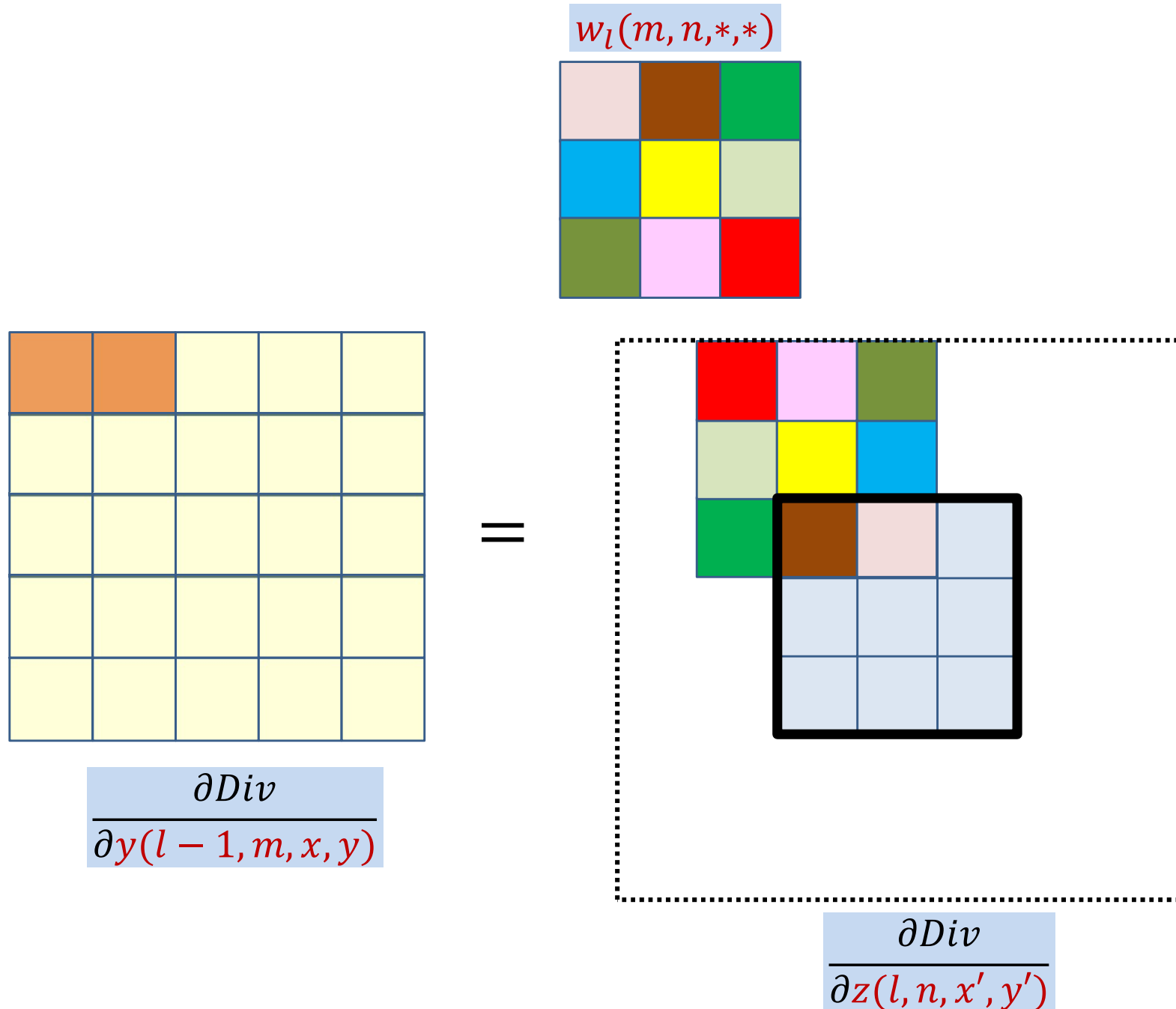
$\frac{\partial Div}{\partial y(l-1, m, x, y)}$

$\frac{\partial Div}{\partial z(l, n, x', y')}$

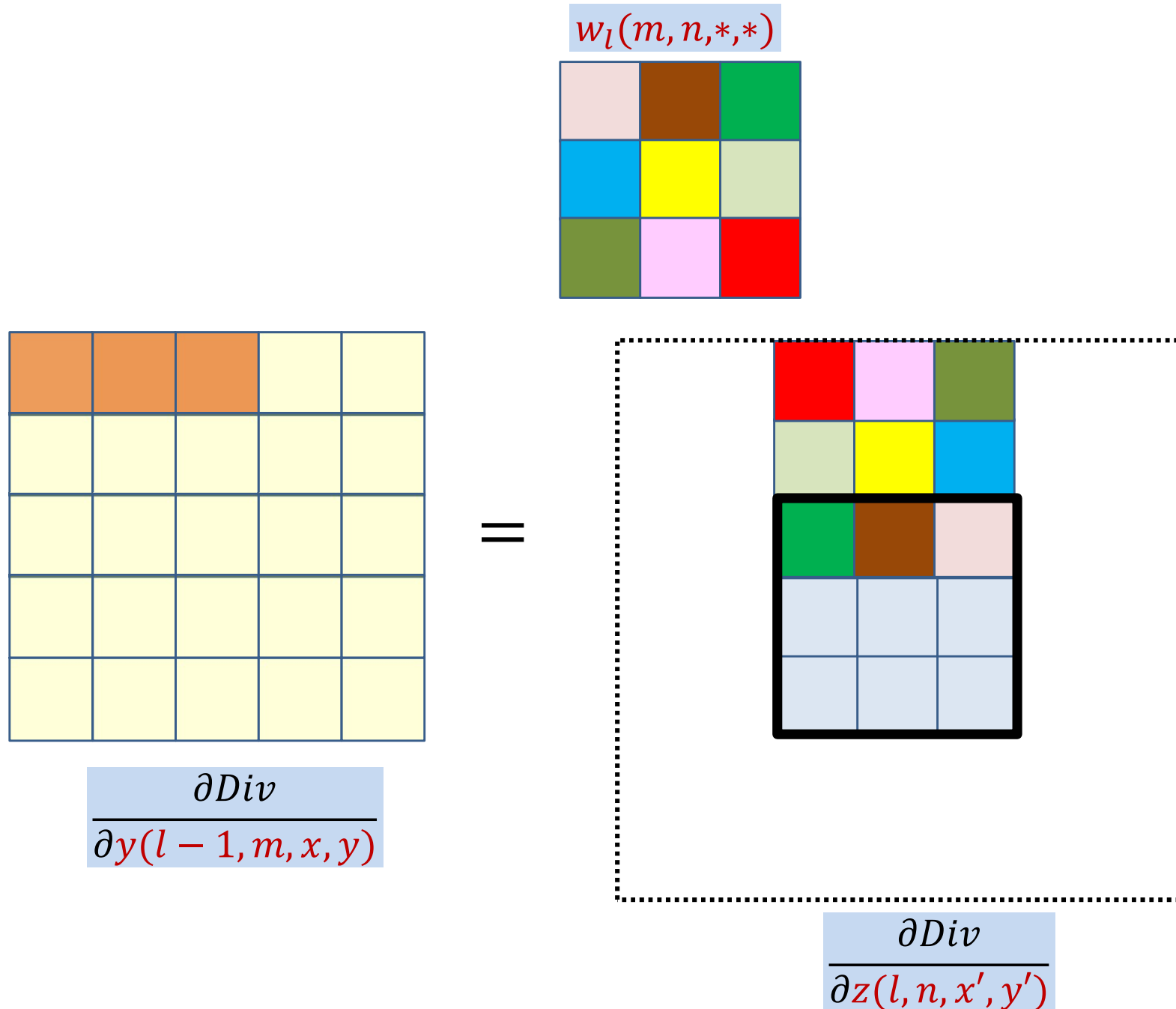
Derivative at $Y(l-1, m)$ from a single $Z(l, n)$ map



Derivative at $Y(l-1, m)$ from a single $Z(l, n)$ map

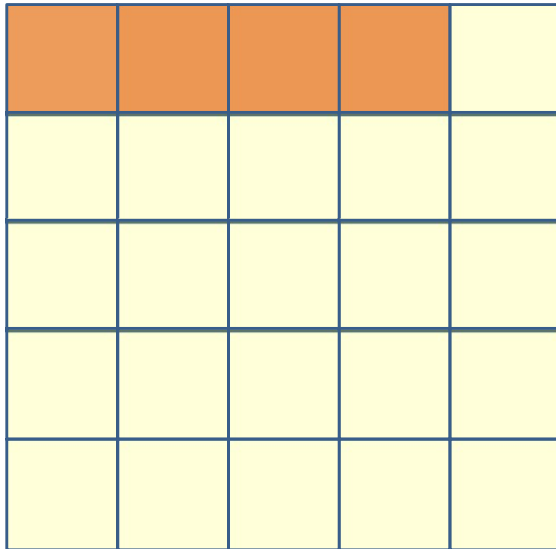
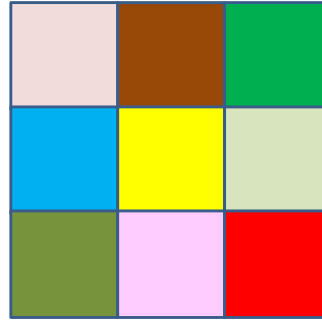


Derivative at $Y(l-1, m)$ from a single $Z(l, n)$ map



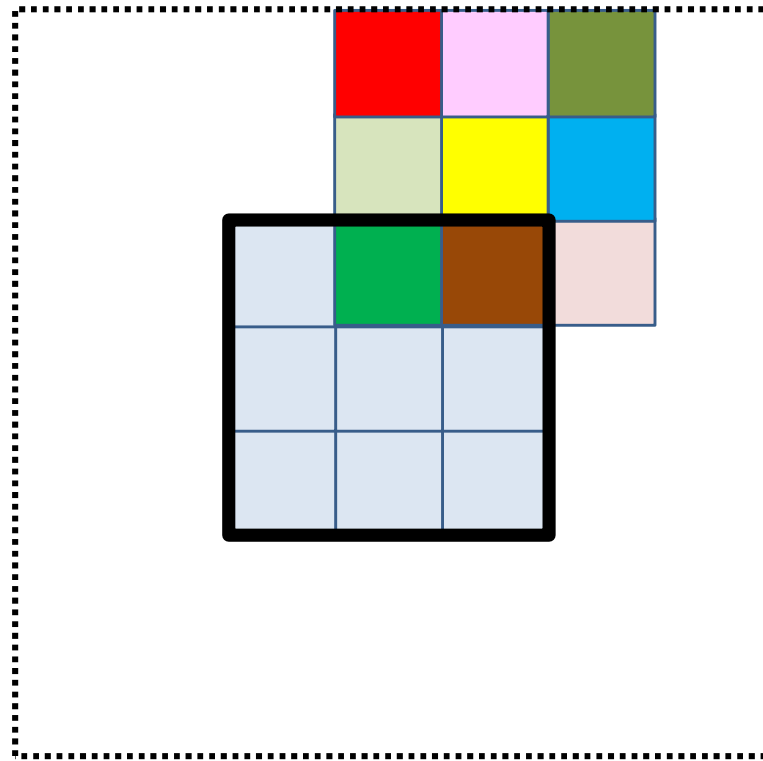
Derivative at $Y(l-1, m)$ from a single $Z(l, n)$ map

$w_l(m, n, *, *)$



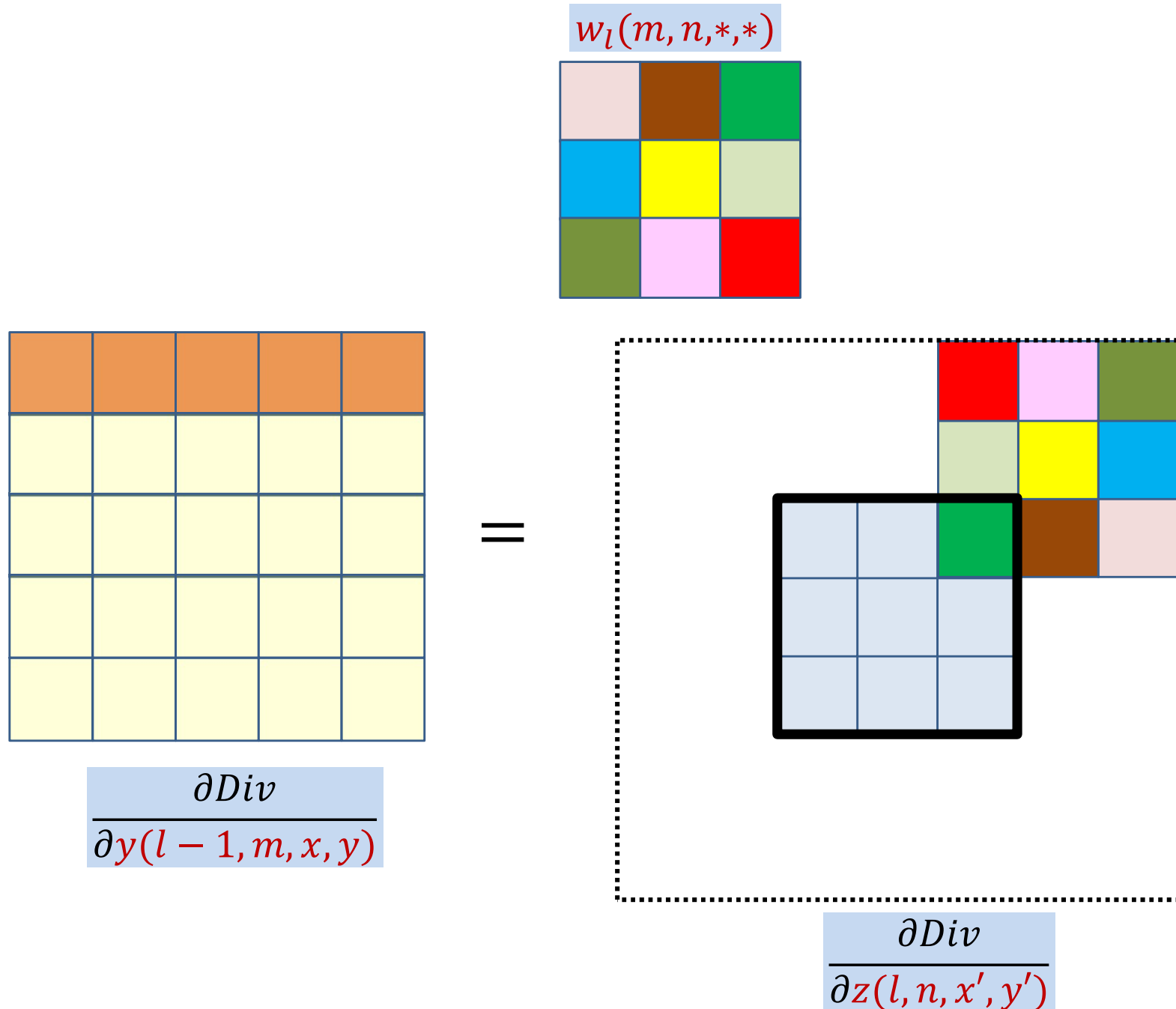
$\frac{\partial Div}{\partial y(l-1, m, x, y)}$

=

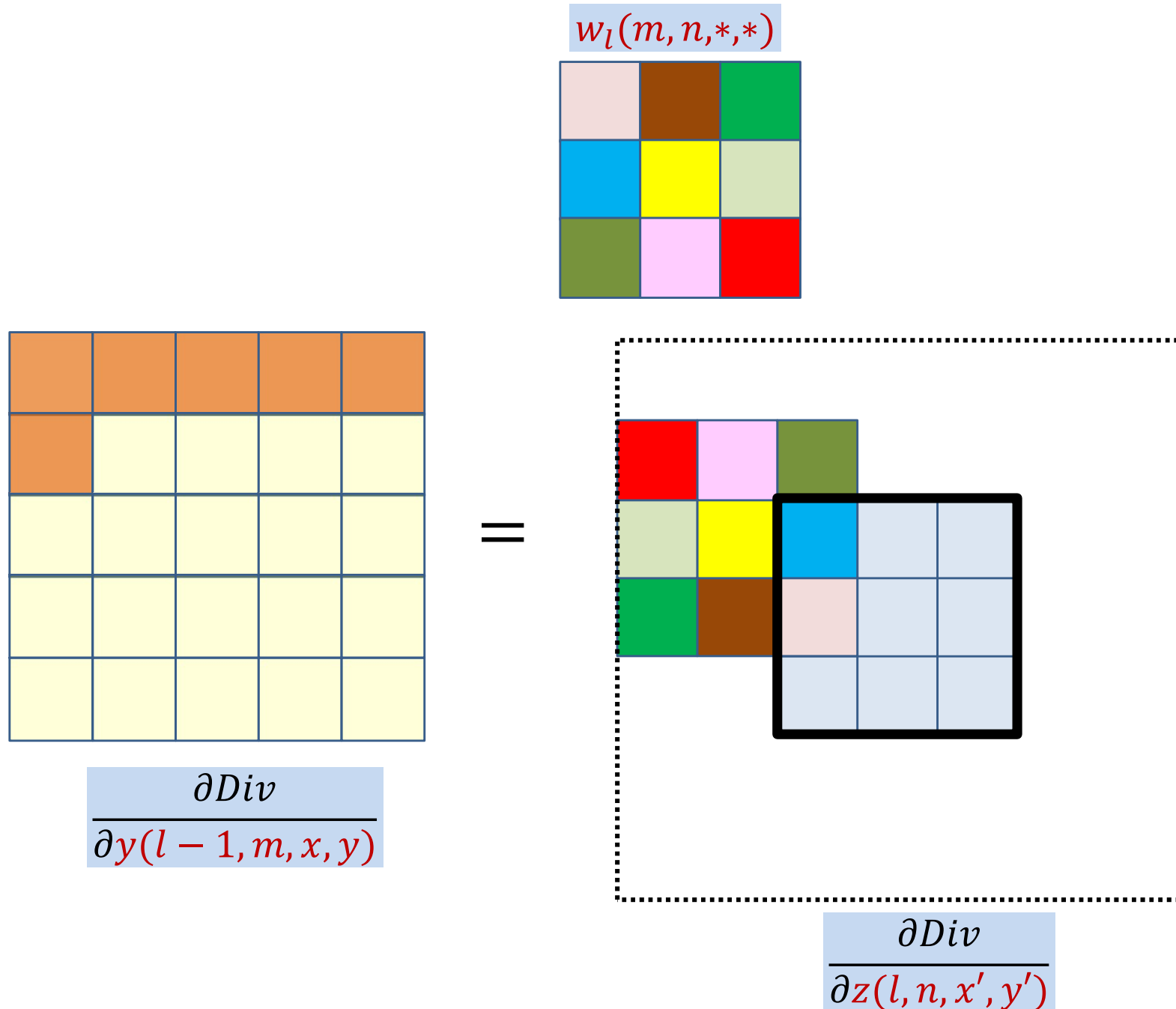


$\frac{\partial Div}{\partial z(l, n, x', y')}$

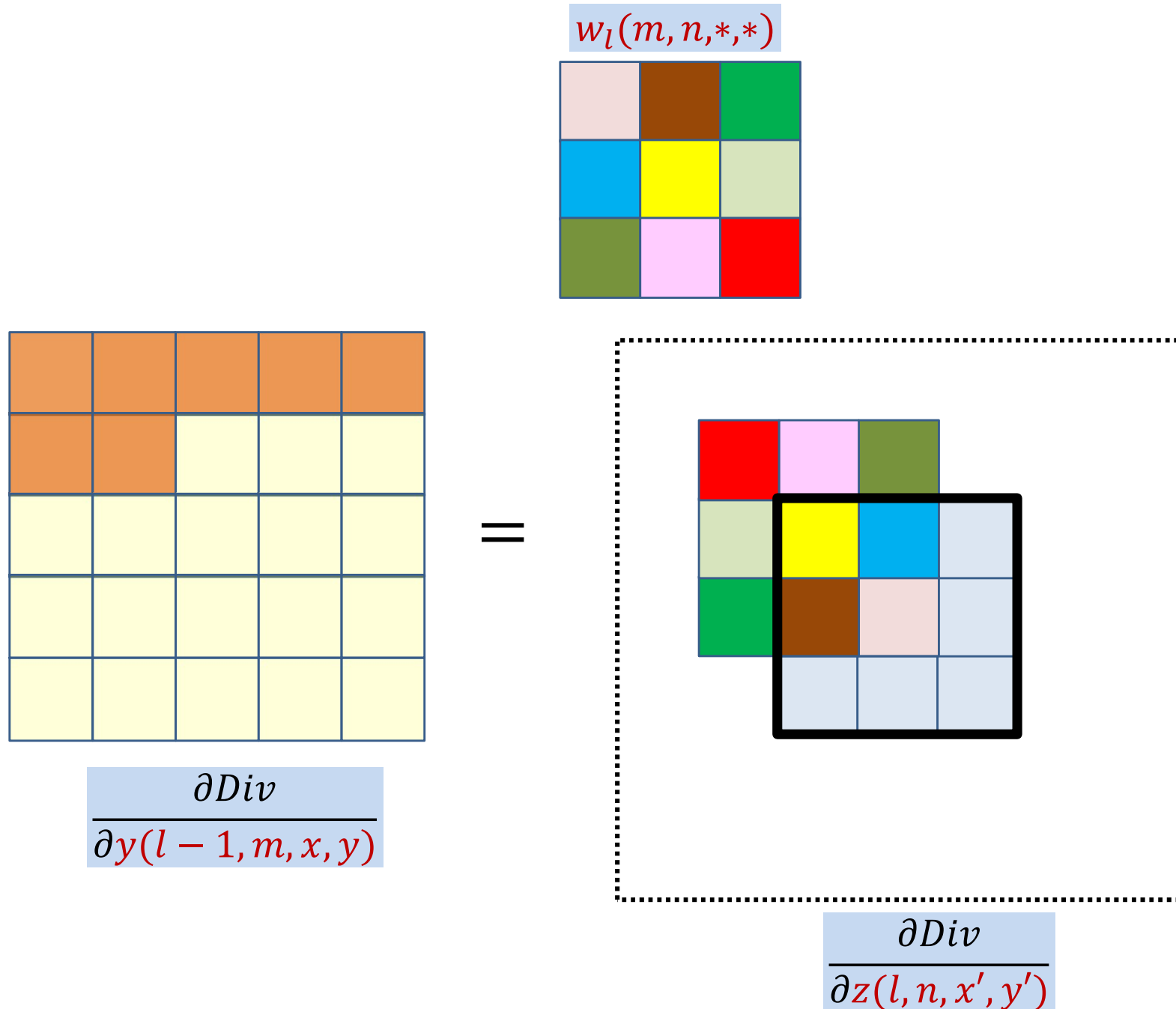
Derivative at $Y(l-1, m)$ from a single $Z(l, n)$ map



Derivative at $Y(l-1, m)$ from a single $Z(l, n)$ map

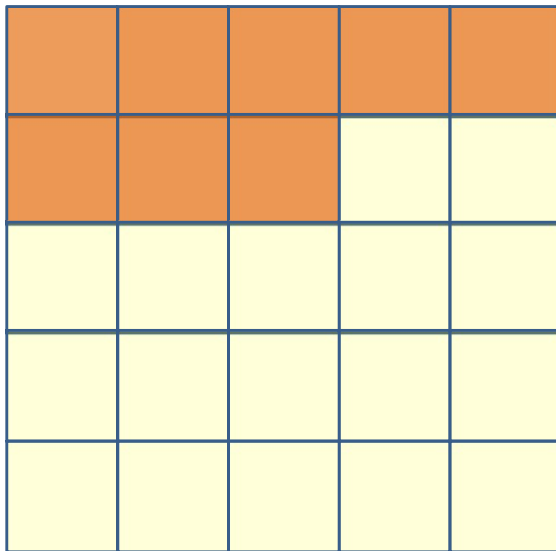
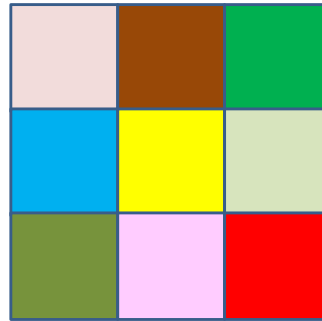


Derivative at $Y(l-1, m)$ from a single $Z(l, n)$ map



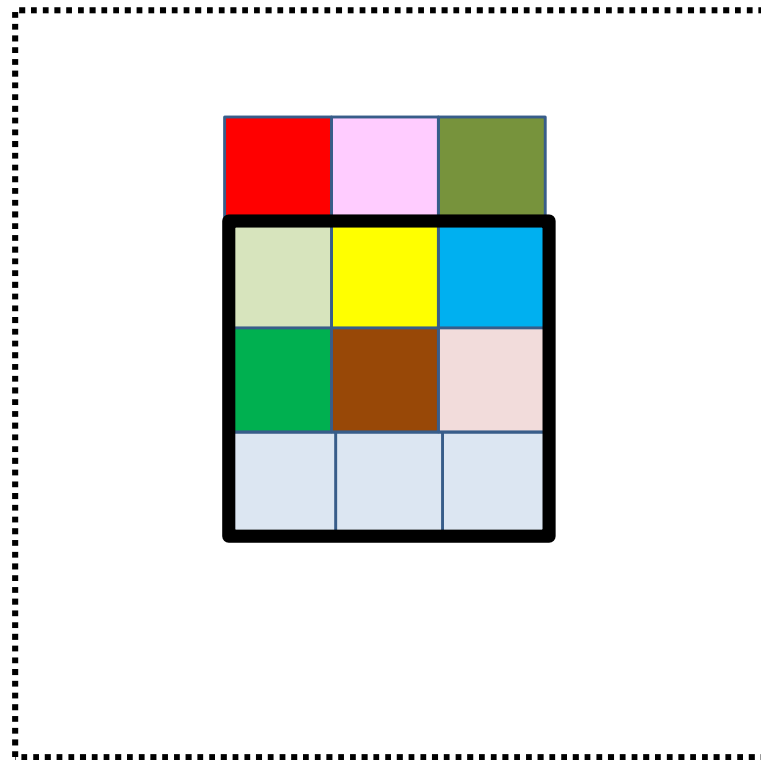
Derivative at $Y(l-1, m)$ from a single $Z(l, n)$ map

$w_l(m, n, *, *)$



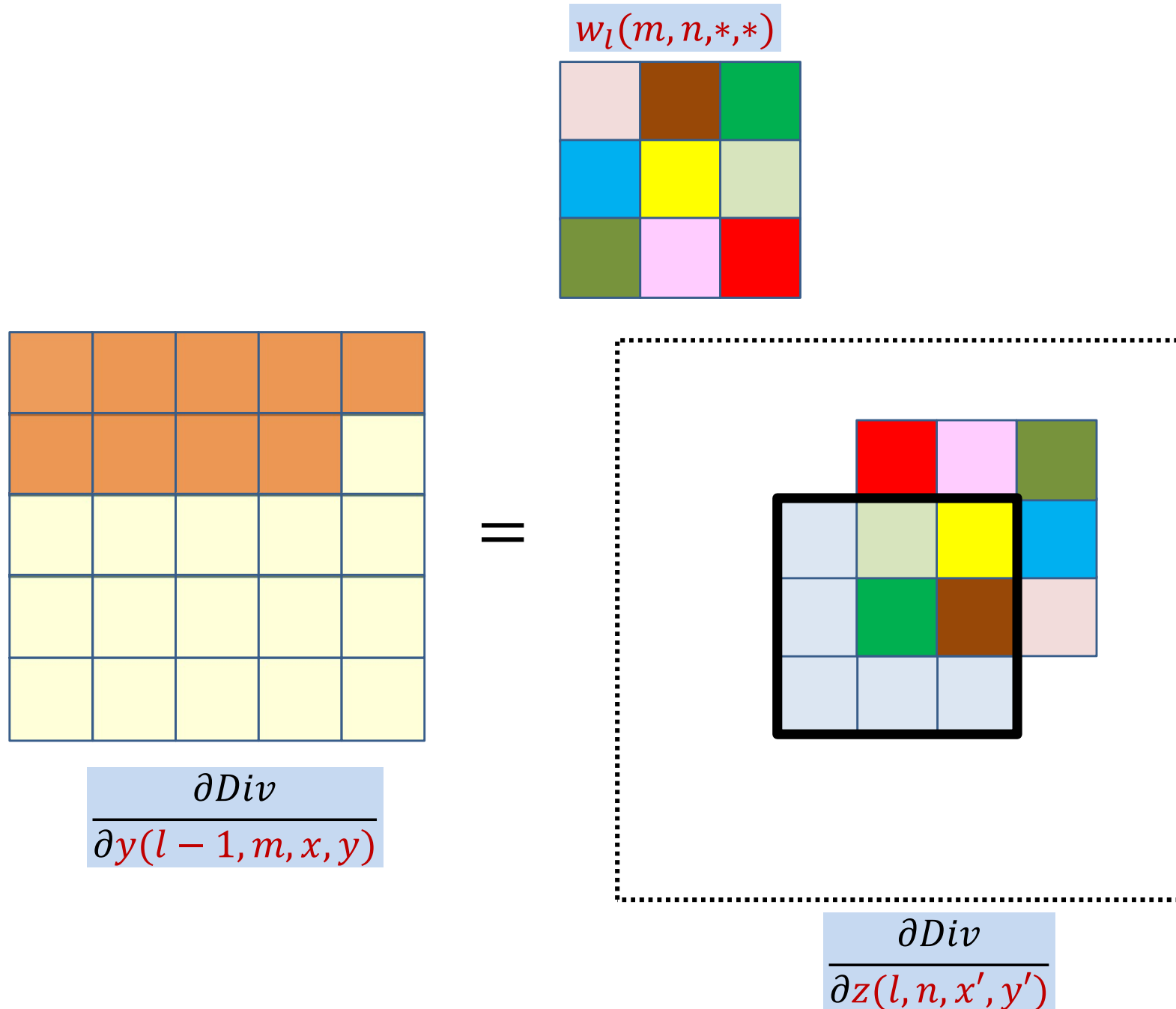
$\frac{\partial Div}{\partial y(l-1, m, x, y)}$

=

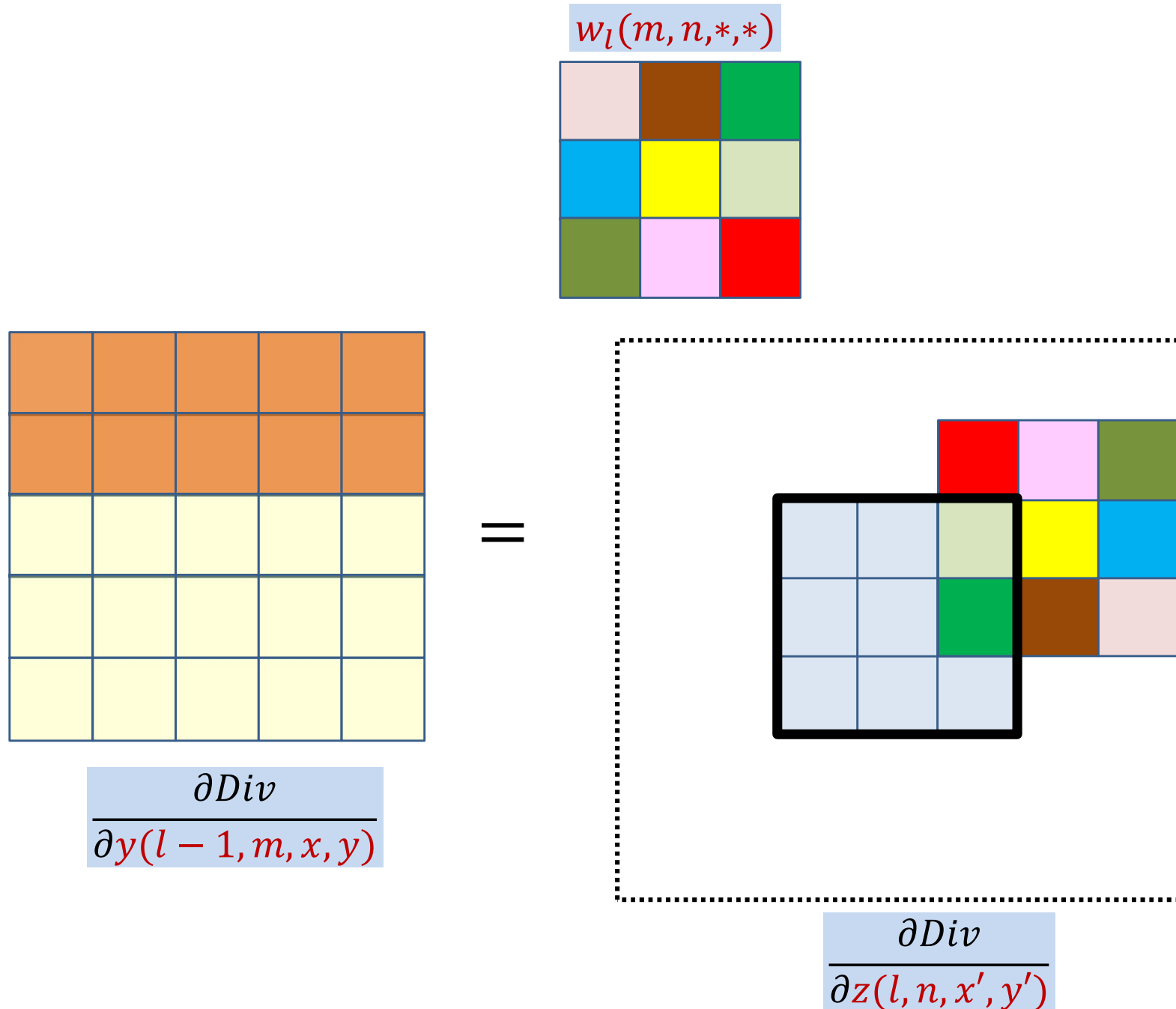


$\frac{\partial Div}{\partial z(l, n, x', y')}$

Derivative at $Y(l-1, m)$ from a single $Z(l, n)$ map

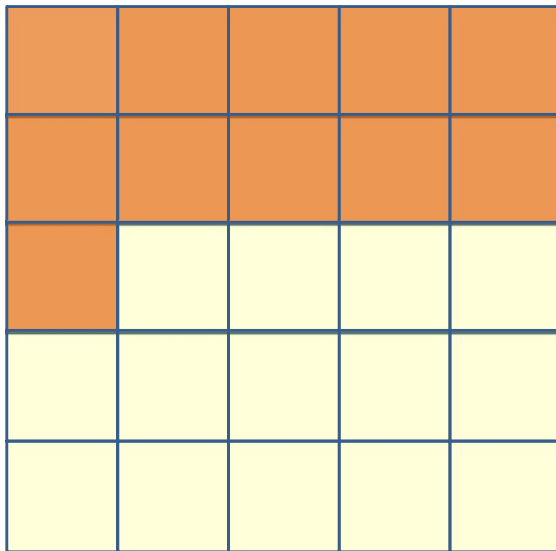
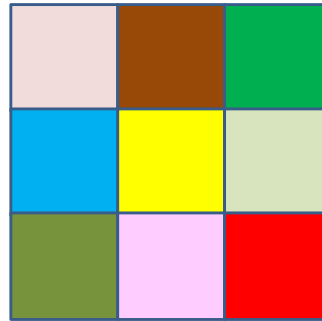


Derivative at $Y(l-1, m)$ from a single $Z(l, n)$ map



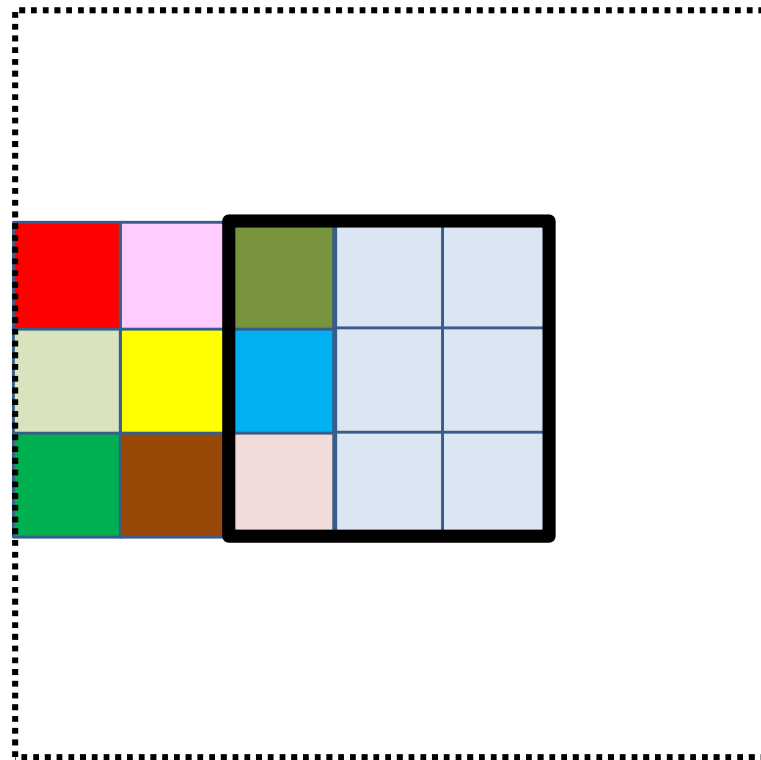
Derivative at $Y(l-1, m)$ from a single $Z(l, n)$ map

$w_l(m, n, *, *)$



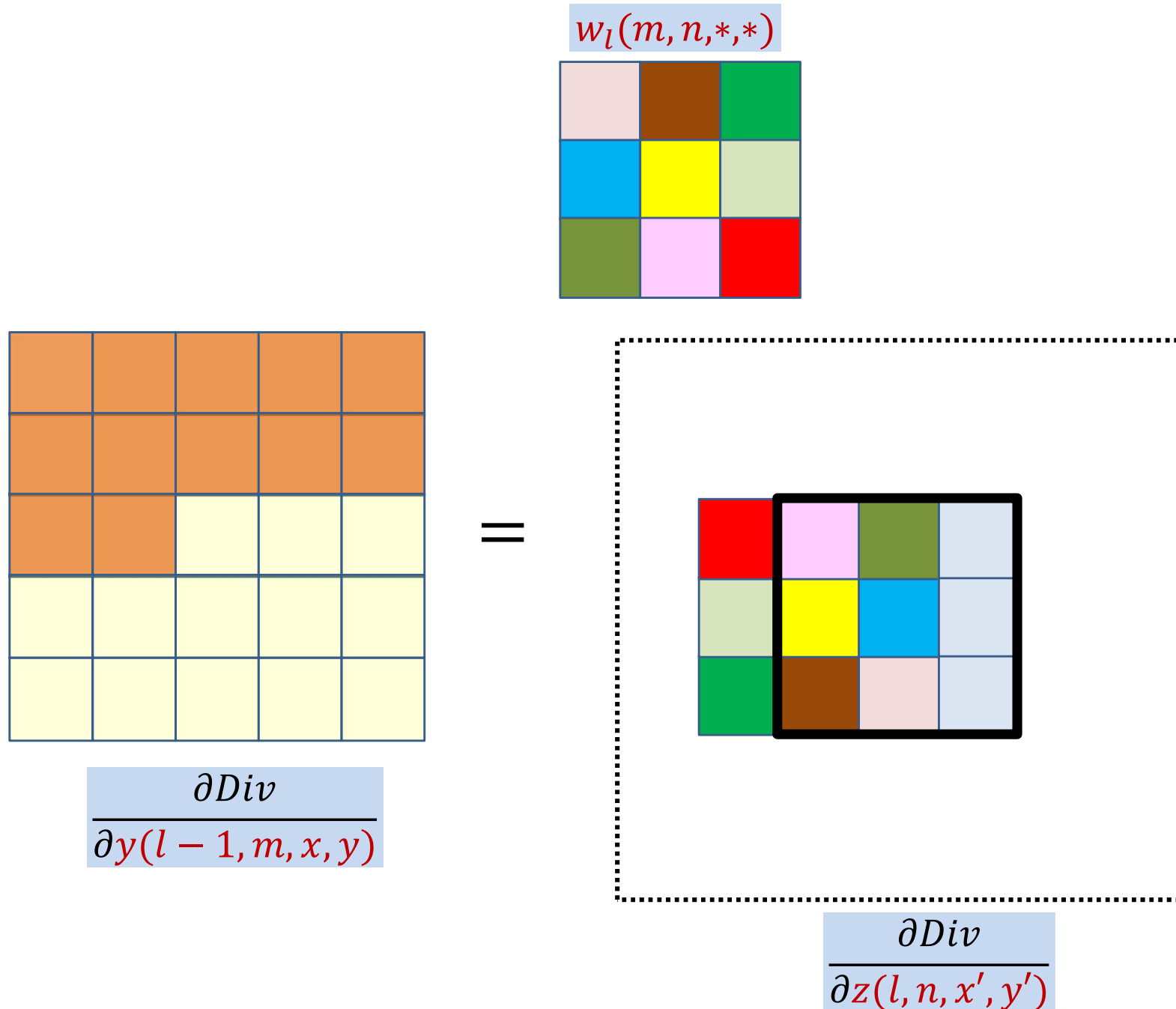
$$\frac{\partial Div}{\partial y(l-1, m, x, y)}$$

=

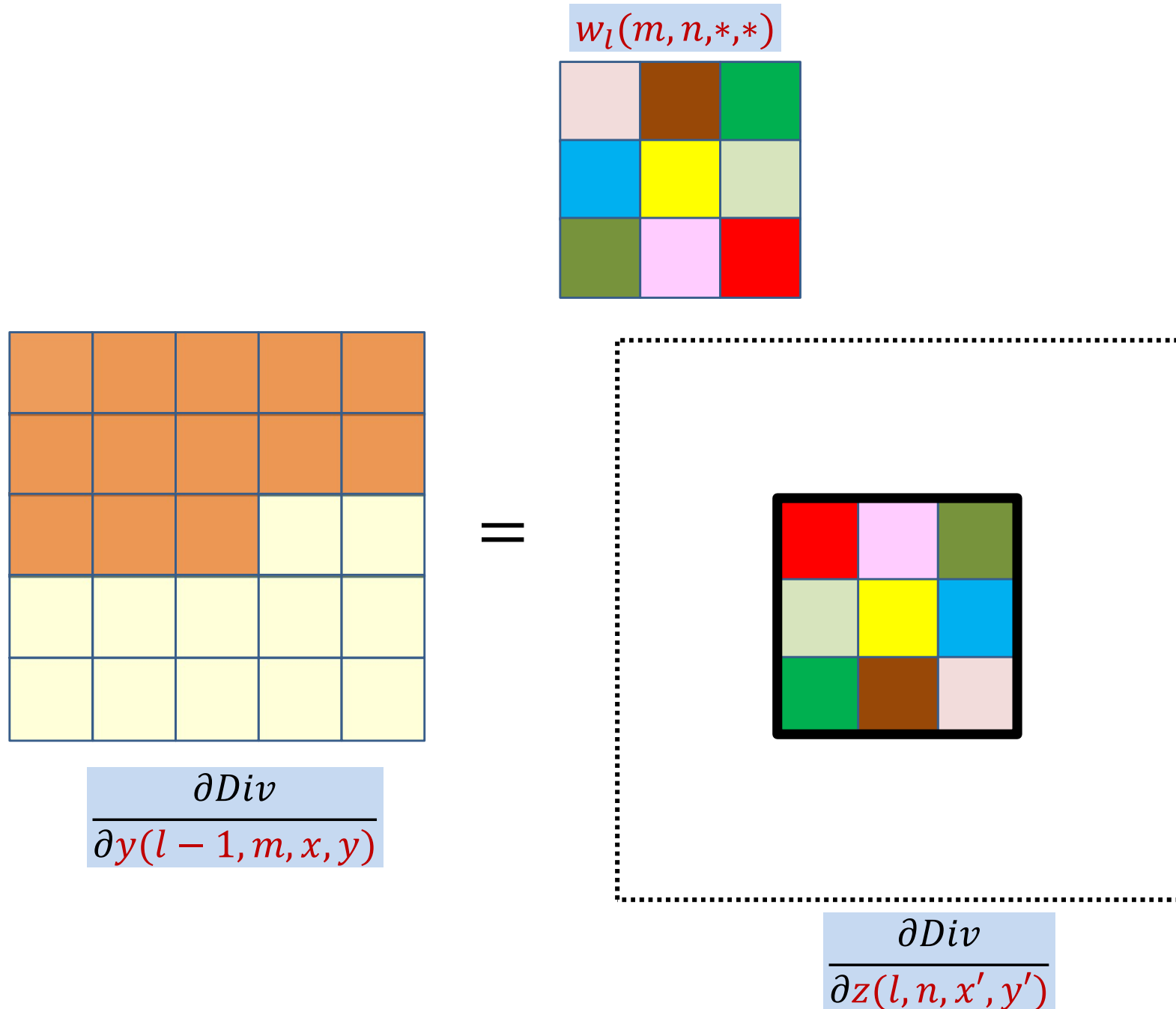


$$\frac{\partial Div}{\partial z(l, n, x', y')}$$

Derivative at $Y(l-1, m)$ from a single $Z(l, n)$ map

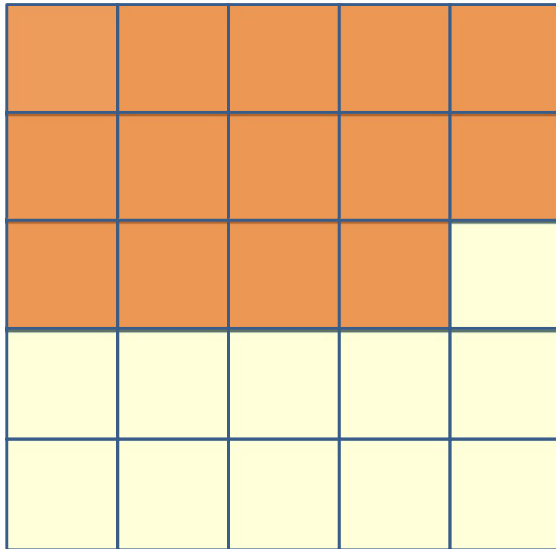
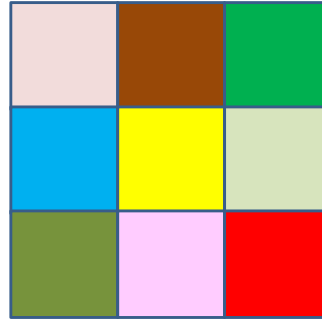


Derivative at $Y(l-1, m)$ from a single $Z(l, n)$ map

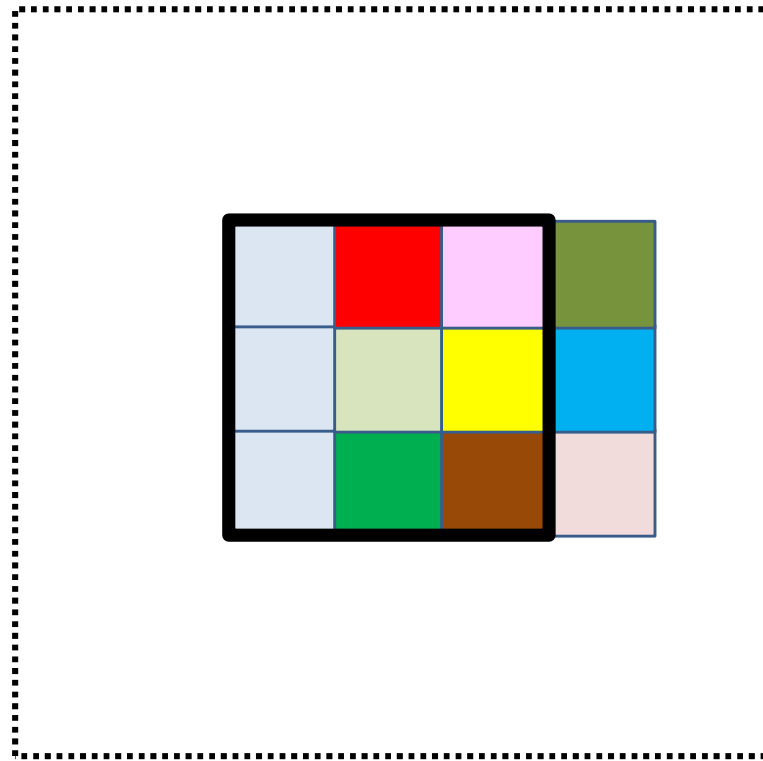


Derivative at $Y(l-1, m)$ from a single $Z(l, n)$ map

$w_l(m, n, *, *)$



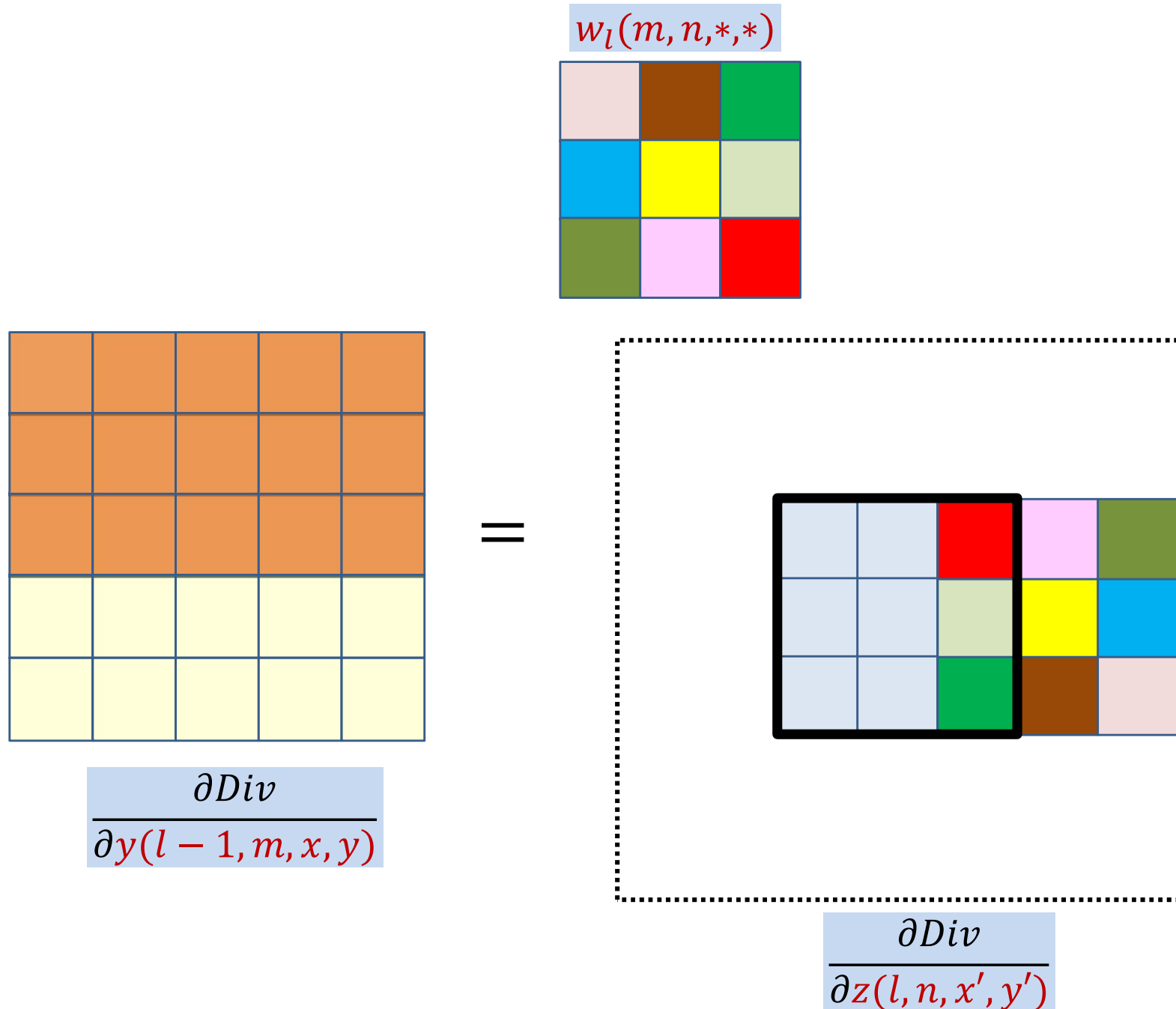
=



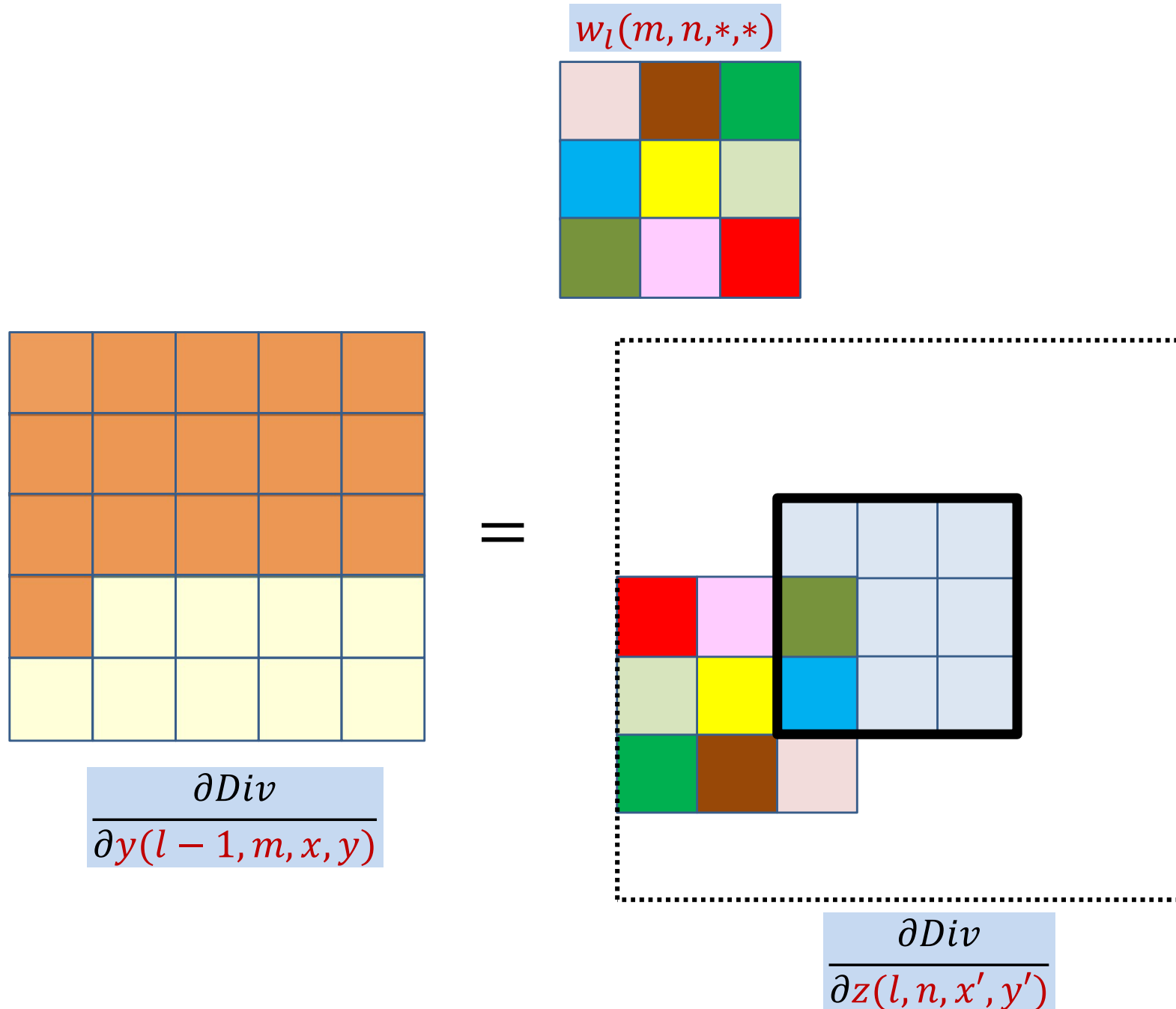
$\frac{\partial Div}{\partial y(l-1, m, x, y)}$

$\frac{\partial Div}{\partial z(l, n, x', y')}$

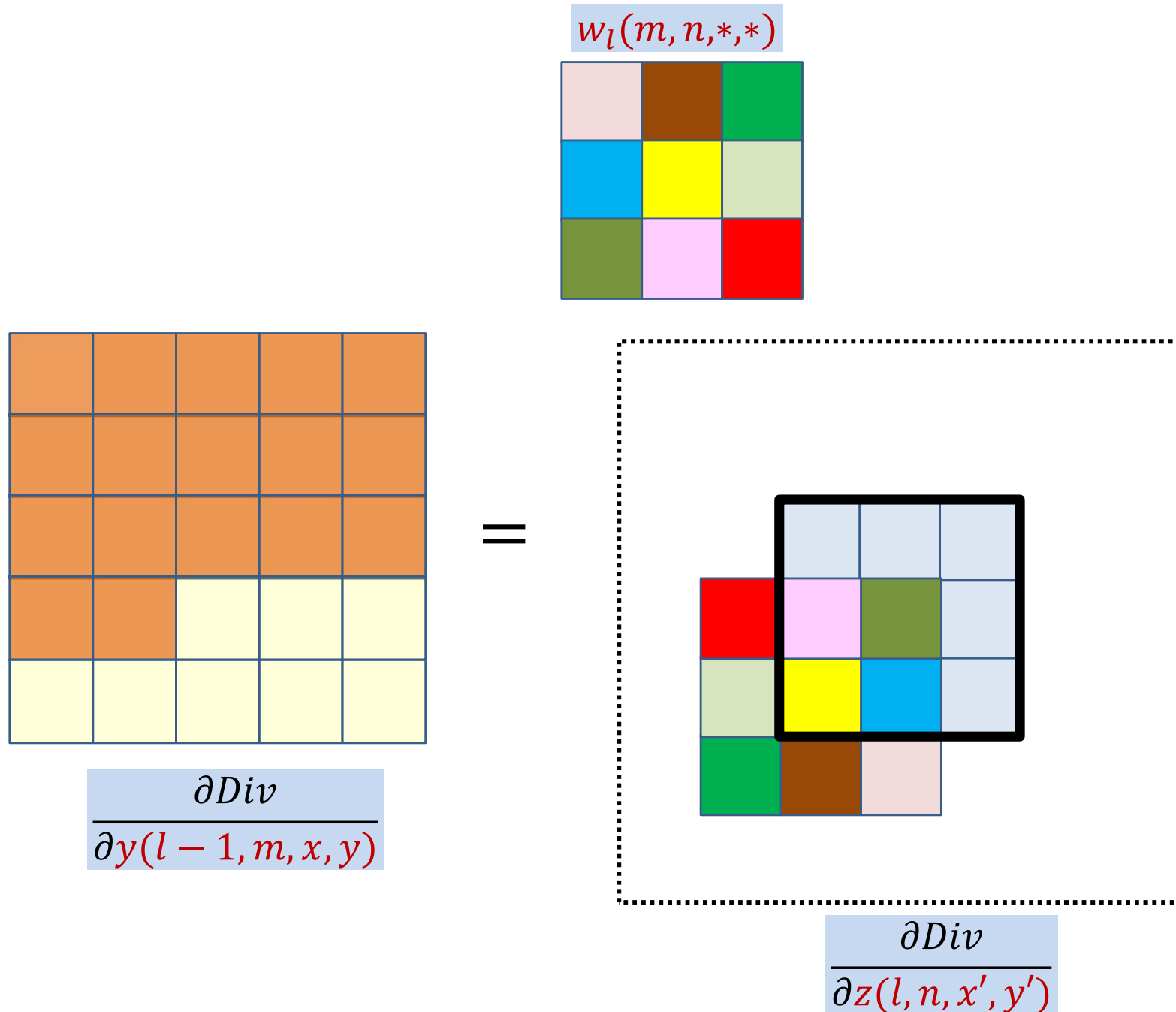
Derivative at $Y(l-1, m)$ from a single $Z(l, n)$ map



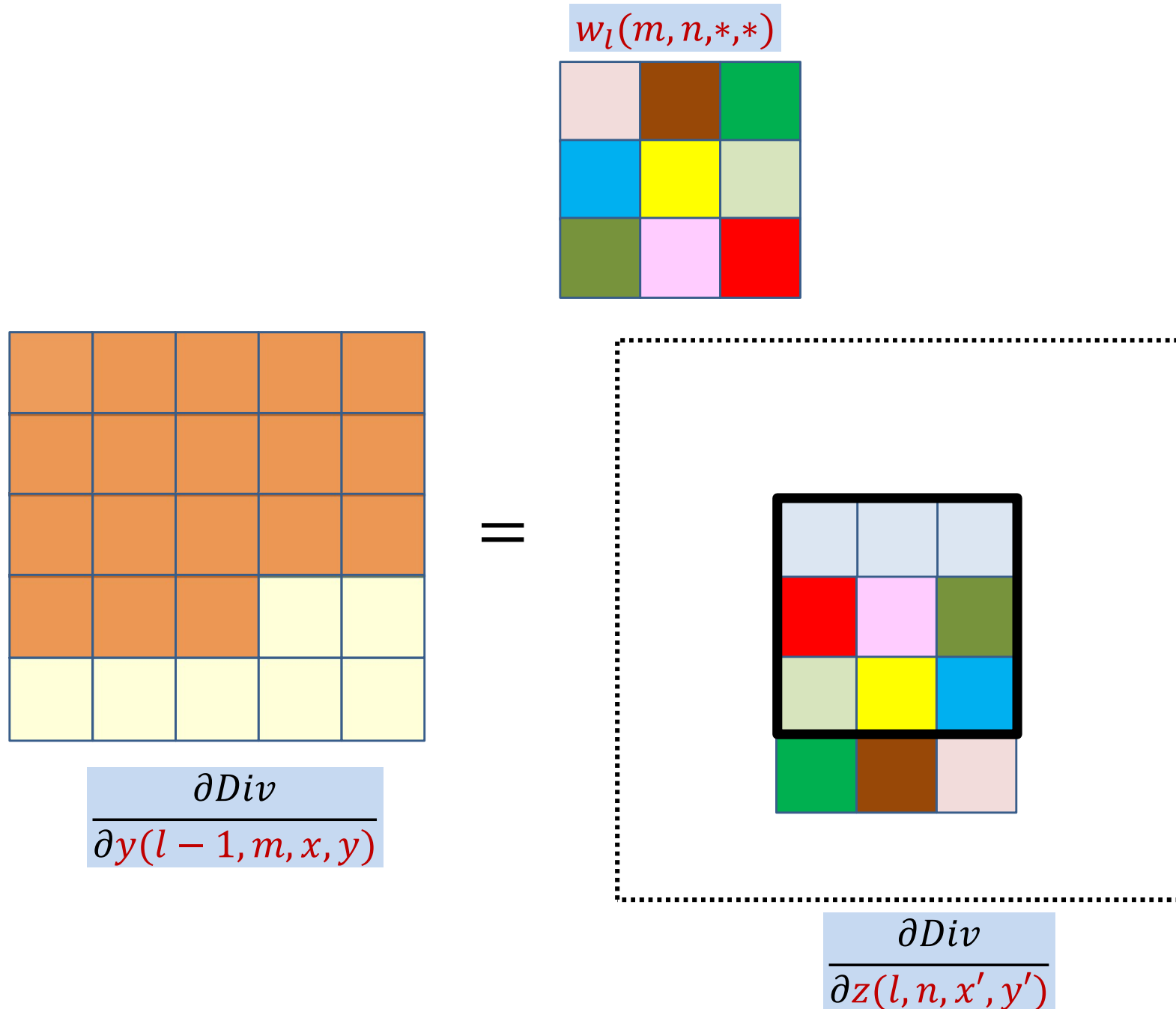
Derivative at $Y(l-1, m)$ from a single $Z(l, n)$ map



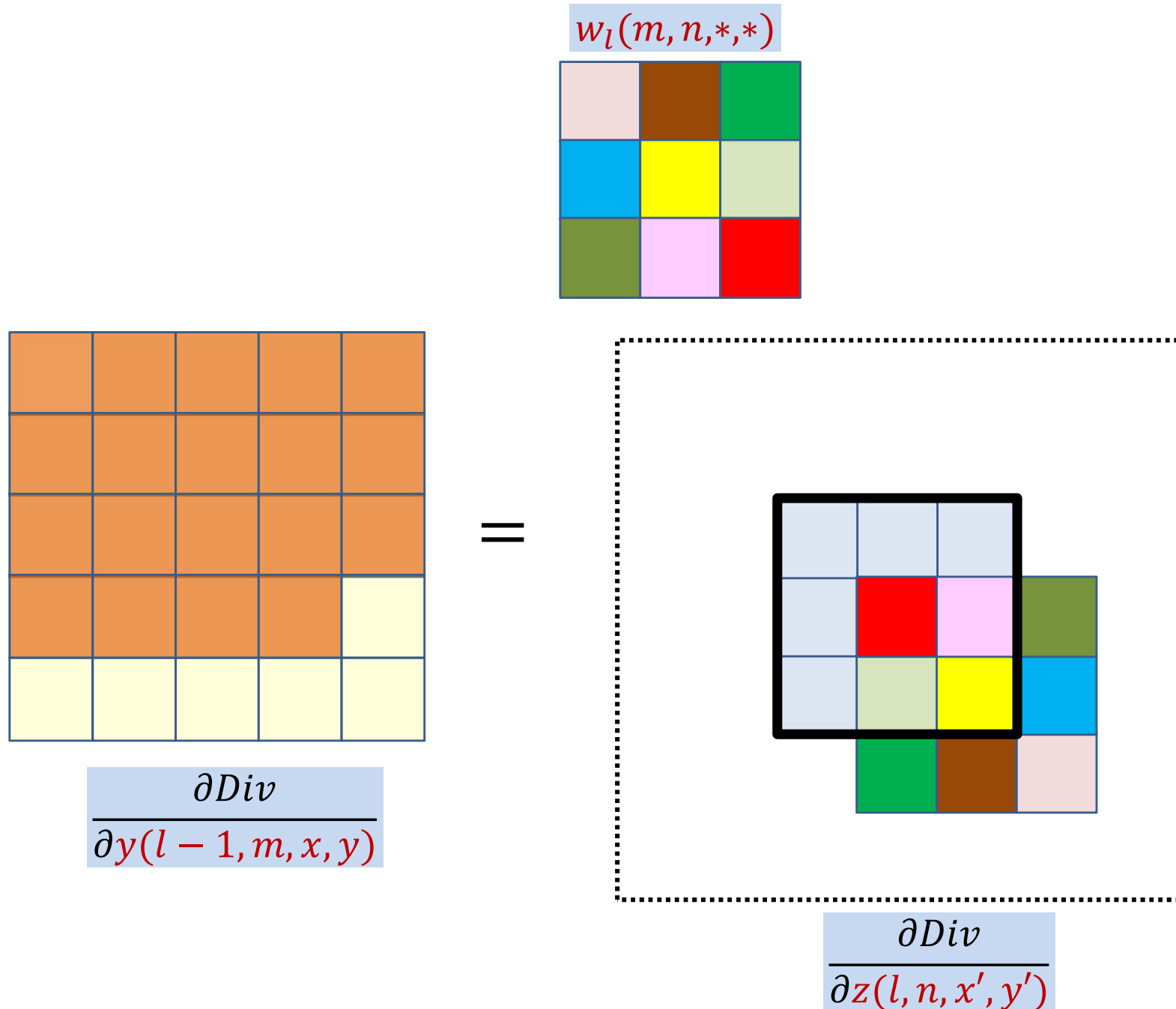
Derivative at $Y(l-1, m)$ from a single $Z(l, n)$ map



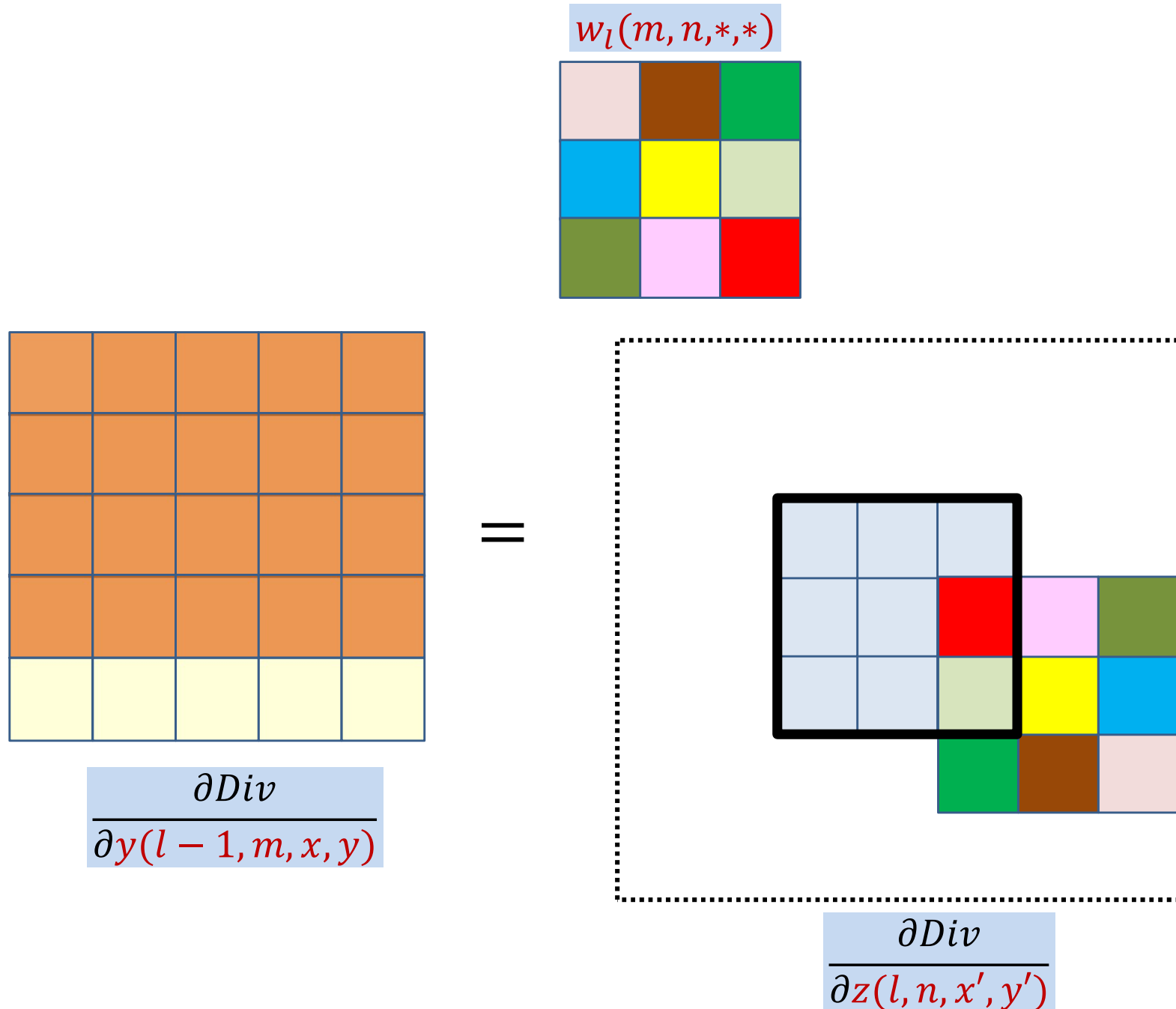
Derivative at $Y(l-1, m)$ from a single $Z(l, n)$ map



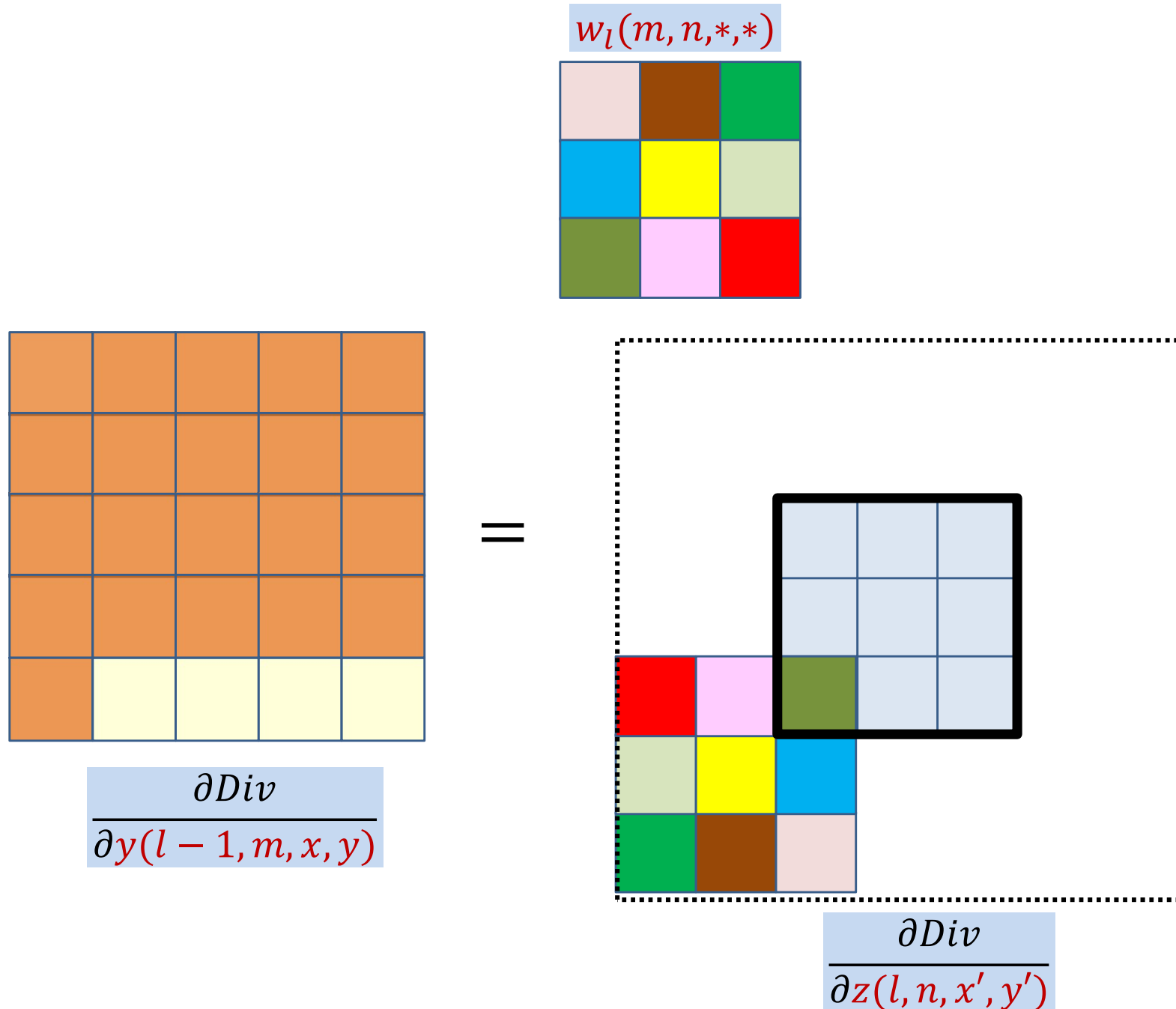
Derivative at $Y(l-1, m)$ from a single $Z(l, n)$ map



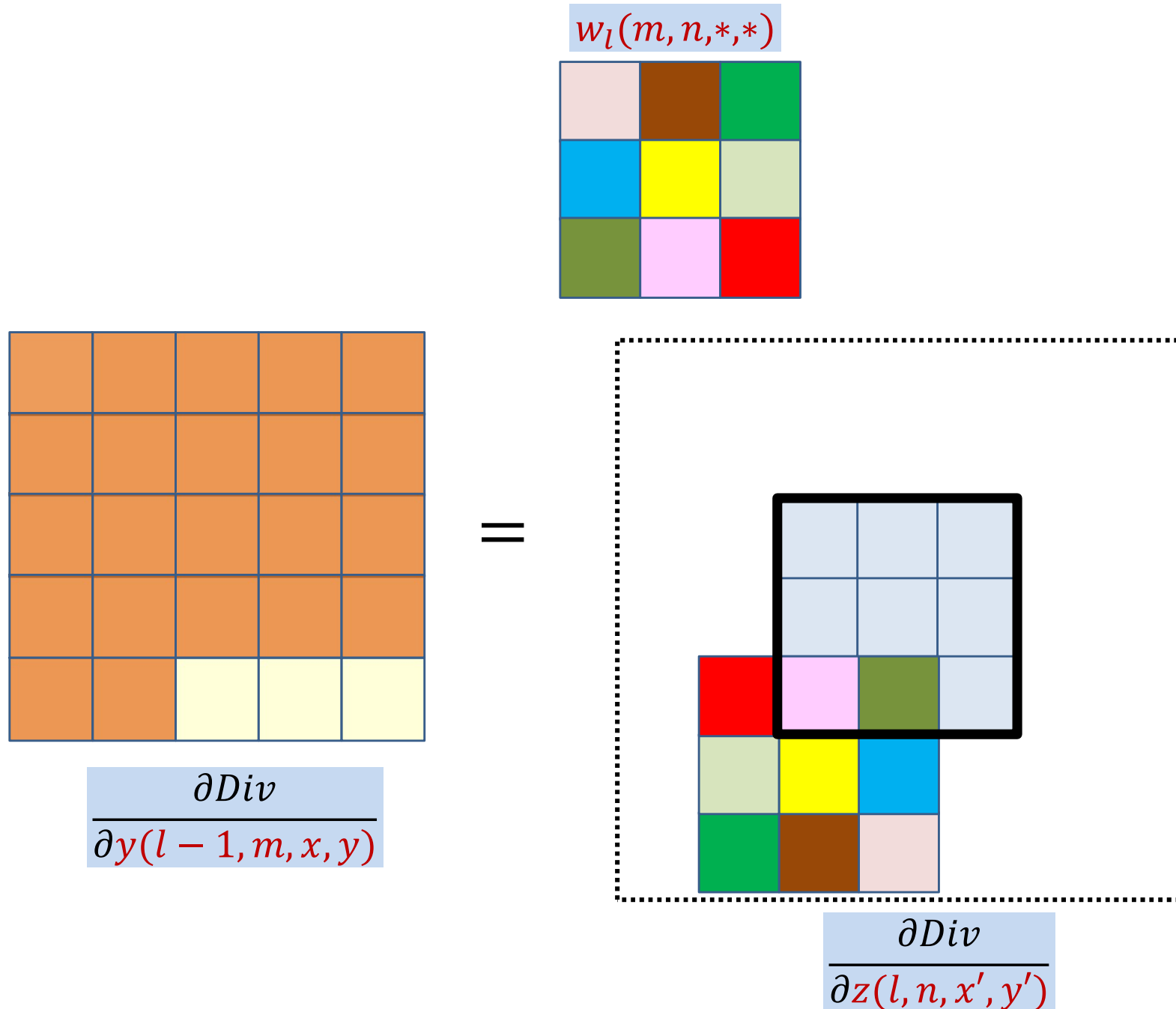
Derivative at $Y(l-1, m)$ from a single $Z(l, n)$ map



Derivative at $Y(l-1, m)$ from a single $Z(l, n)$ map

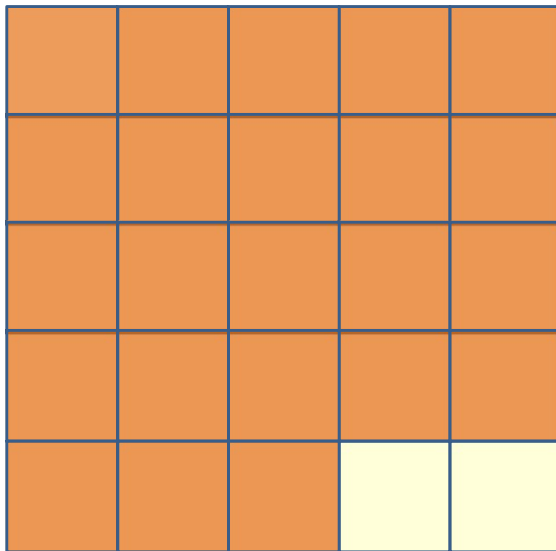
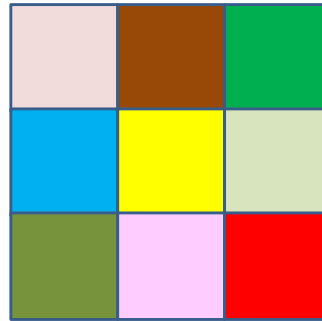


Derivative at $Y(l-1, m)$ from a single $Z(l, n)$ map



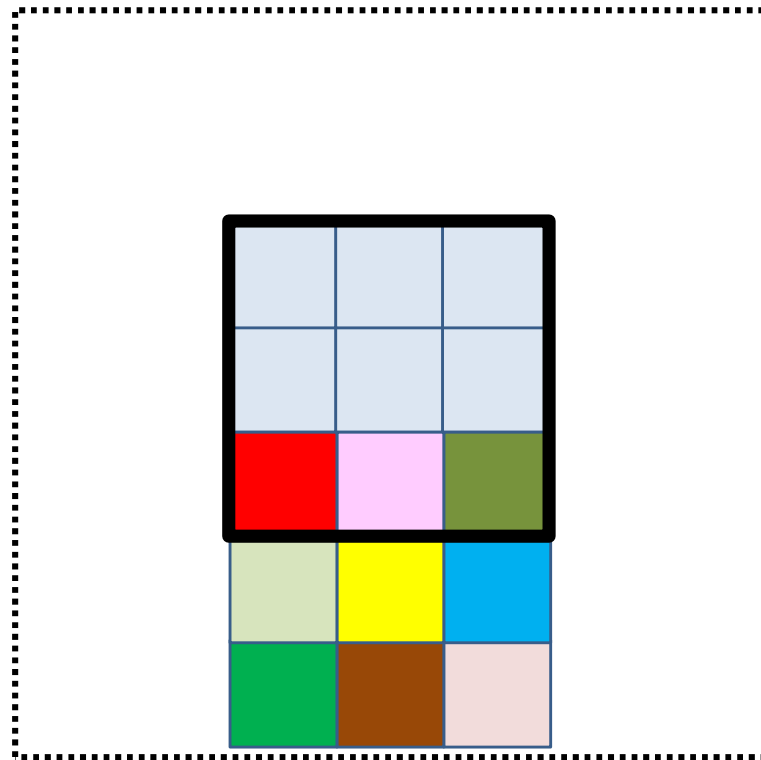
Derivative at $Y(l-1, m)$ from a single $Z(l, n)$ map

$w_l(m, n, *, *)$



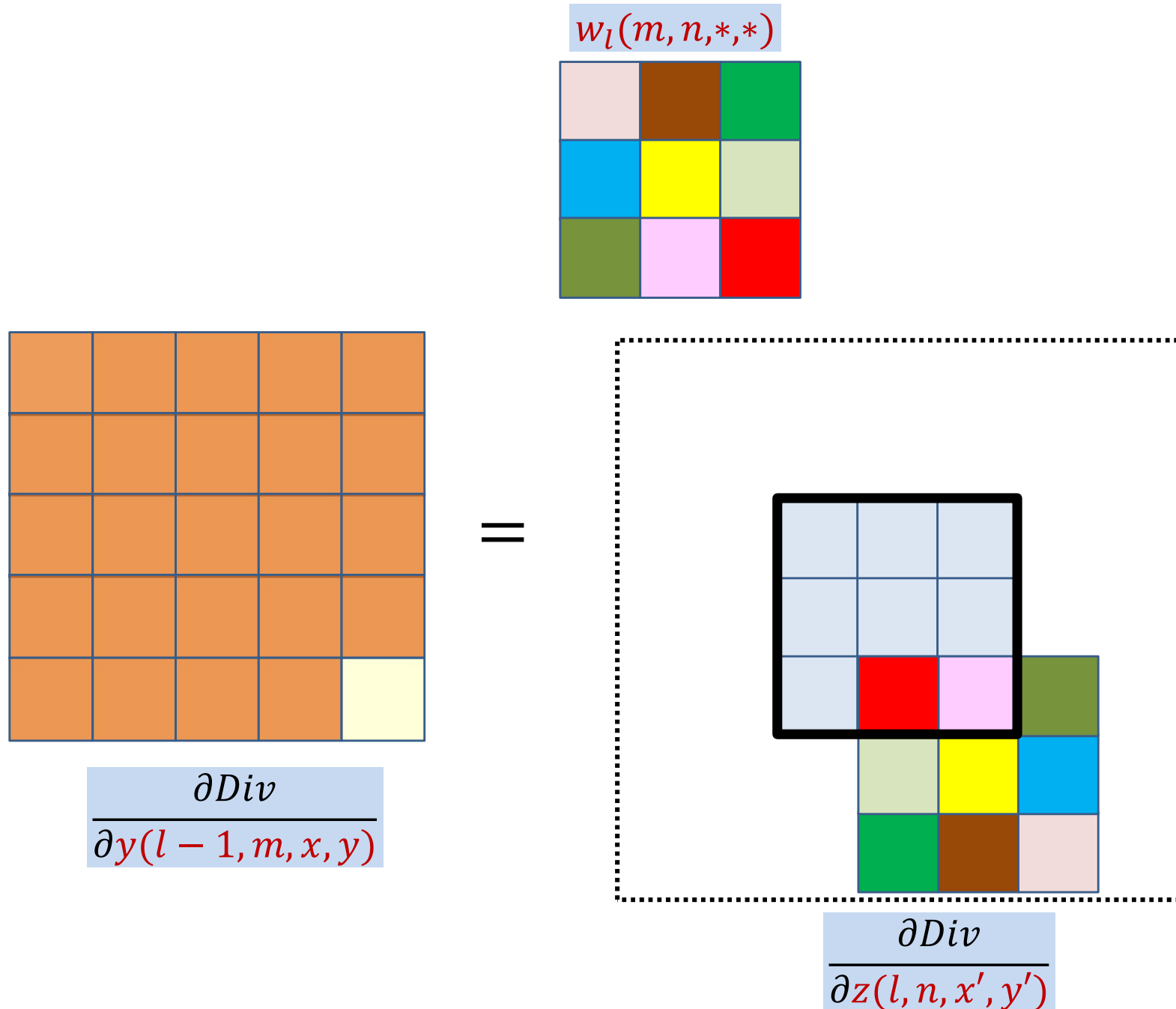
$\frac{\partial Div}{\partial y(l-1, m, x, y)}$

=

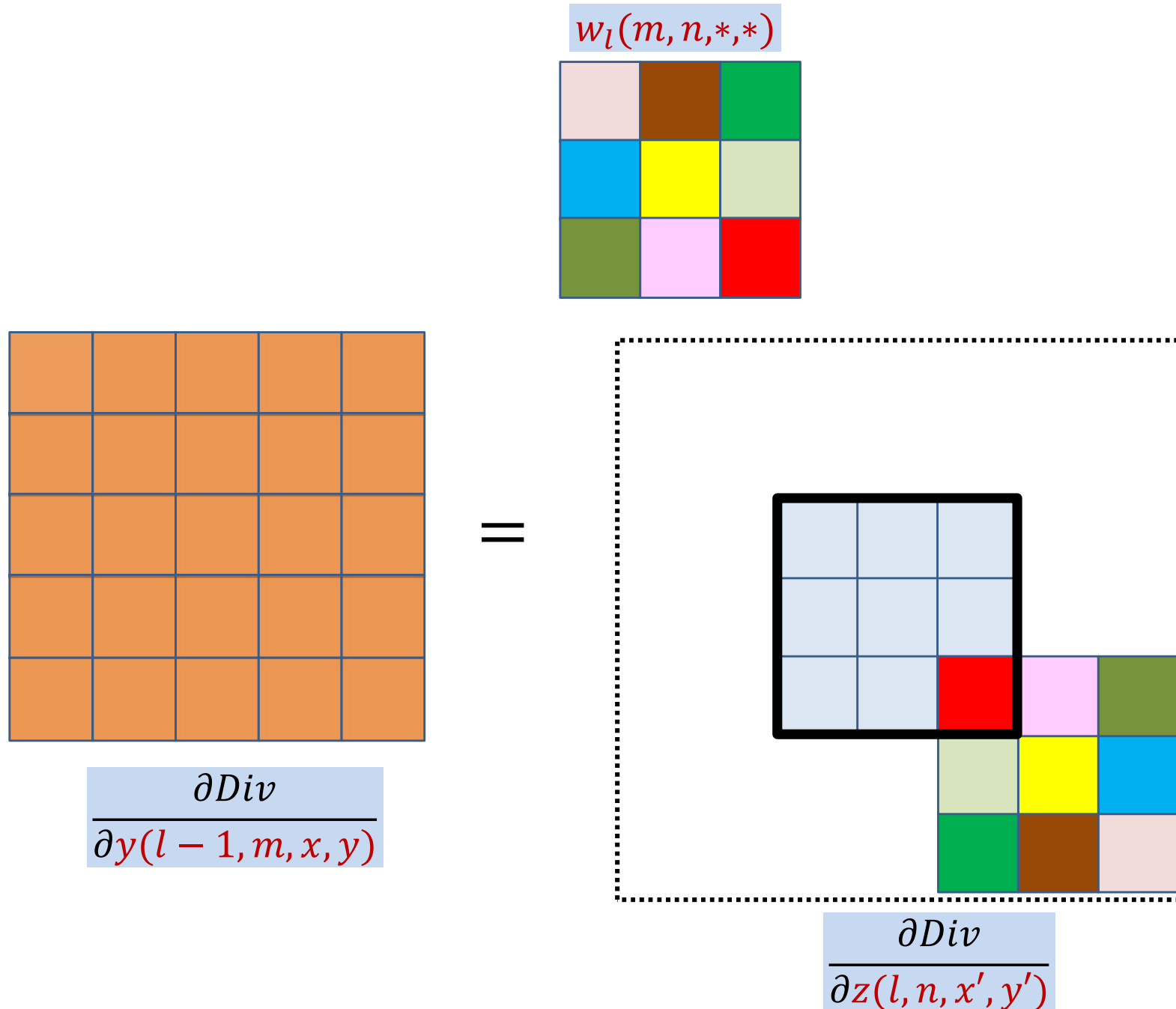


$\frac{\partial Div}{\partial z(l, n, x', y')}$

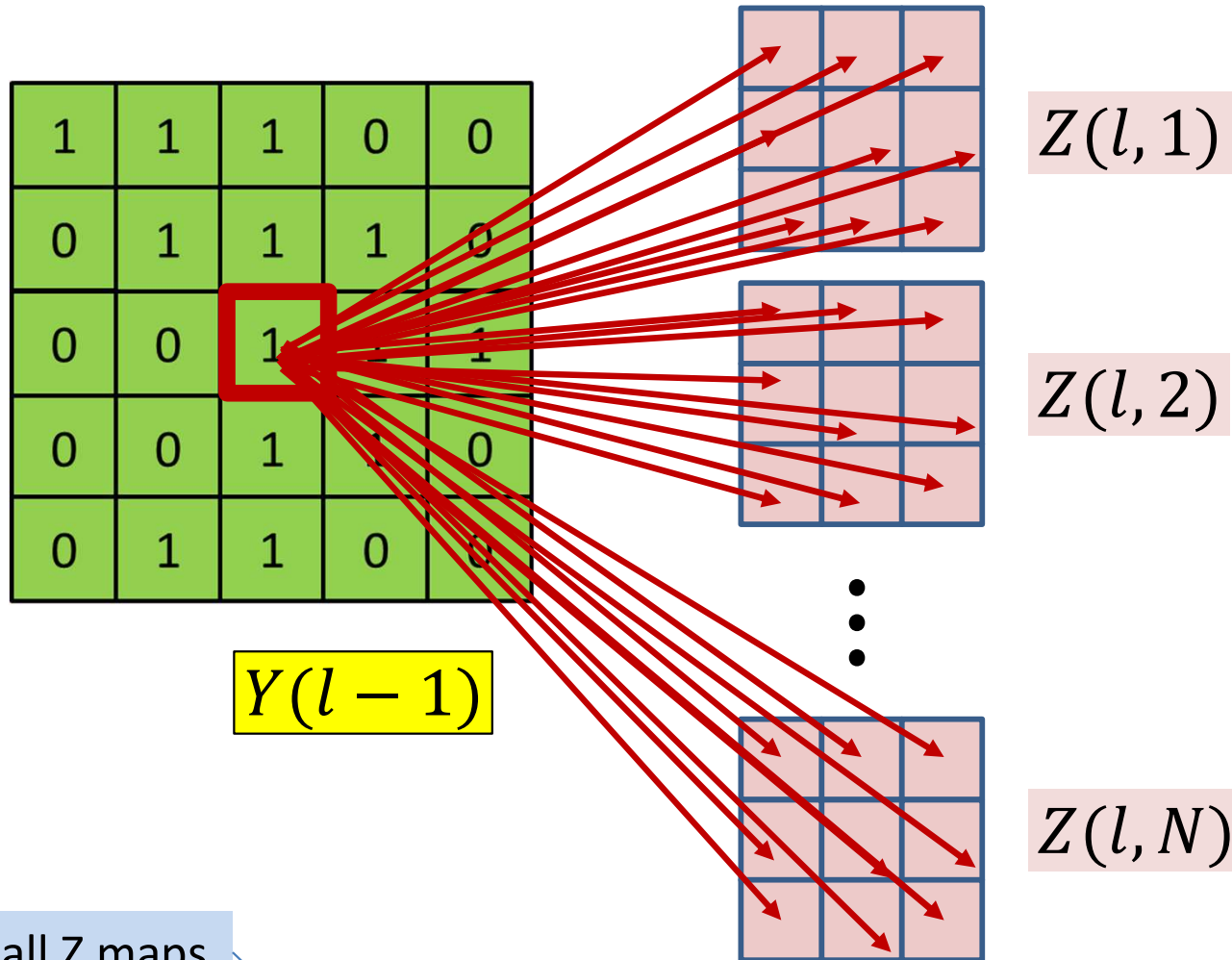
Derivative at $Y(l-1, m)$ from a single $Z(l, n)$ map



Derivative at $Y(l-1, m)$ from a single $Z(l, n)$ map



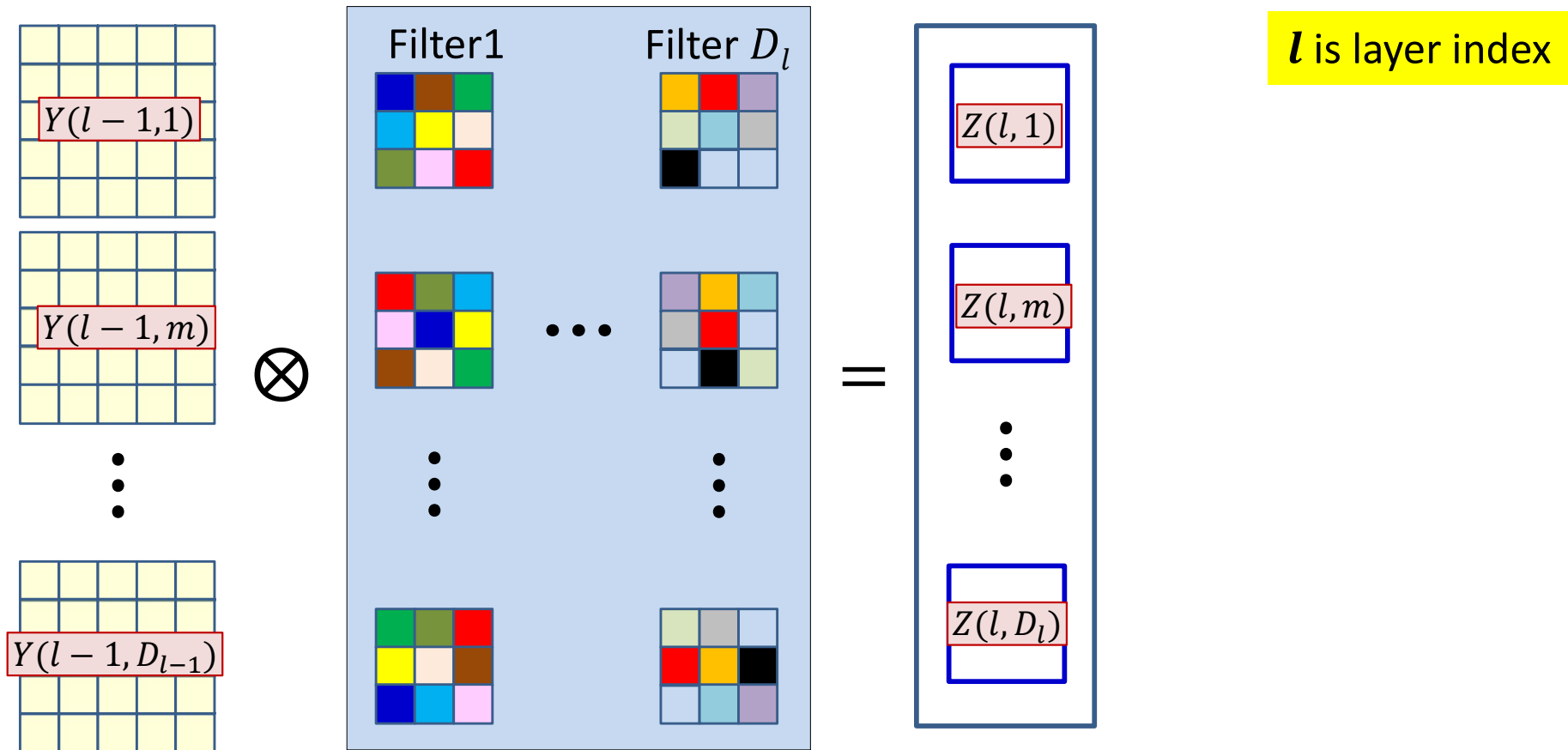
BP: Convolutional layer



Summing over all Z maps

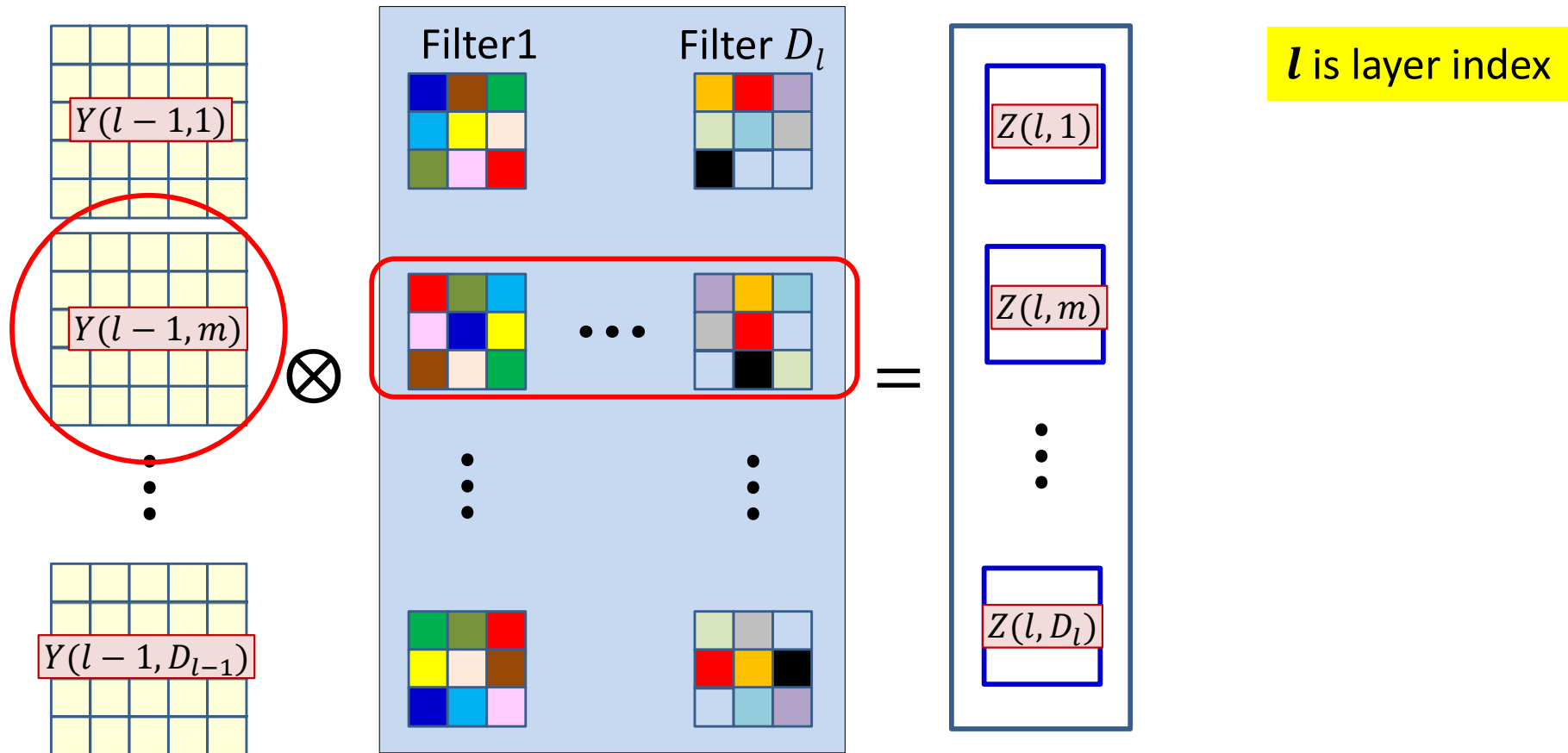
$$\frac{dDiv}{dY(l-1, m, x, y)} = \sum_n \sum_{x', y'} \frac{dDiv}{dz(l, n, x', y')} w_l(m, n, x - x', y - y')$$

The actual convolutions



- The D_l affine maps are produced by convolving with D_l filters

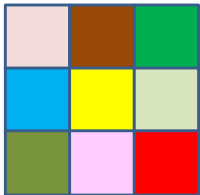
The actual convolutions



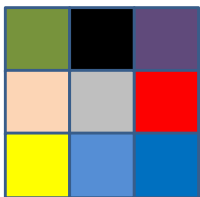
- The D_l affine maps are produced by convolving with D_l filters
- The m^{th} Y map always convolves the m^{th} plane of the filters
- The derivative for the m^{th} Y map will invoke the m^{th} plane of *all* the filters

$$w_l(m, n, x, y)$$

$n = 1$

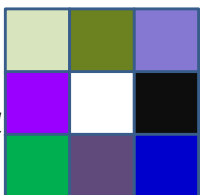


$n = 2$

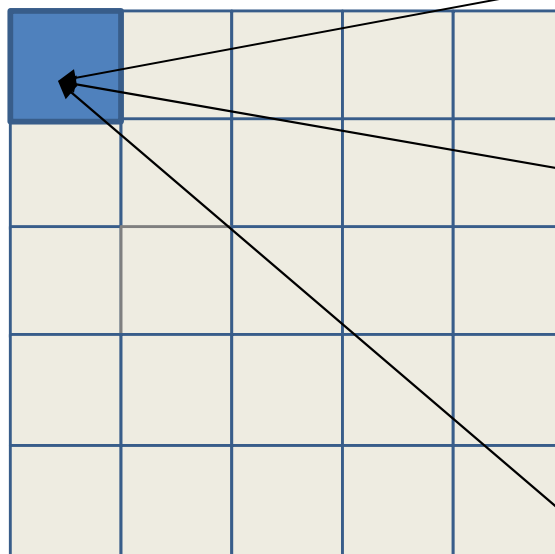


⋮

$n = D_l$

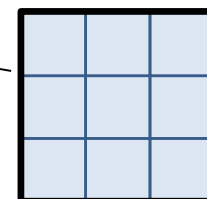
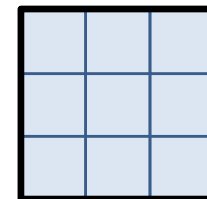


In reality, the derivative at each (x, y) location is obtained from *all* z maps

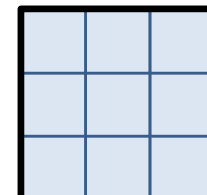


=

$$\frac{\partial Div}{\partial z(l, n, x', y')}$$



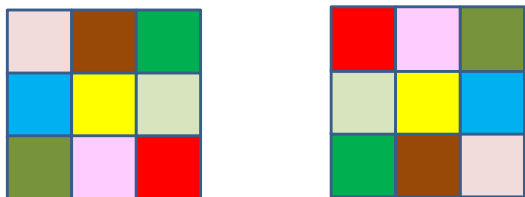
⋮



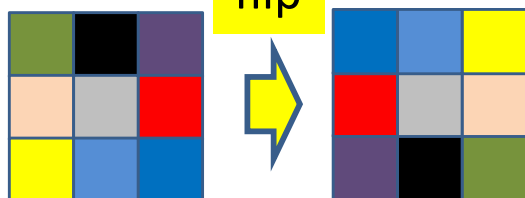
$$\frac{\partial Div}{\partial y(l-1, m, x, y)}$$

$$w_l(m, n, x, y)$$

$n = 1$



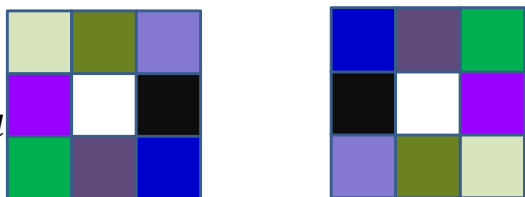
$n = 2$



⋮

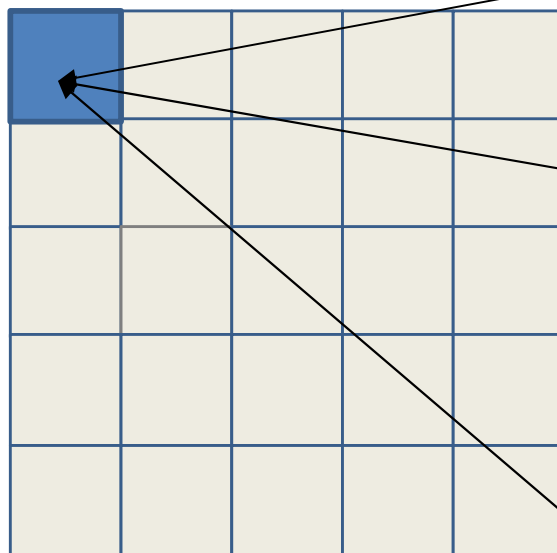
⋮

$n = D_l$



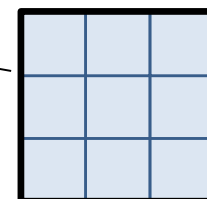
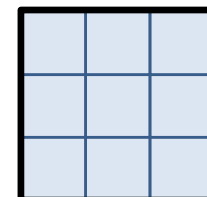
$$w_l(m, n, K + 1 - x, K + 1 - y)$$

In reality, the derivative at each (x, y) location is obtained from *all* z maps

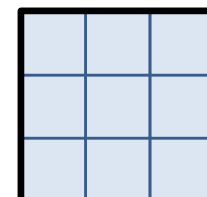


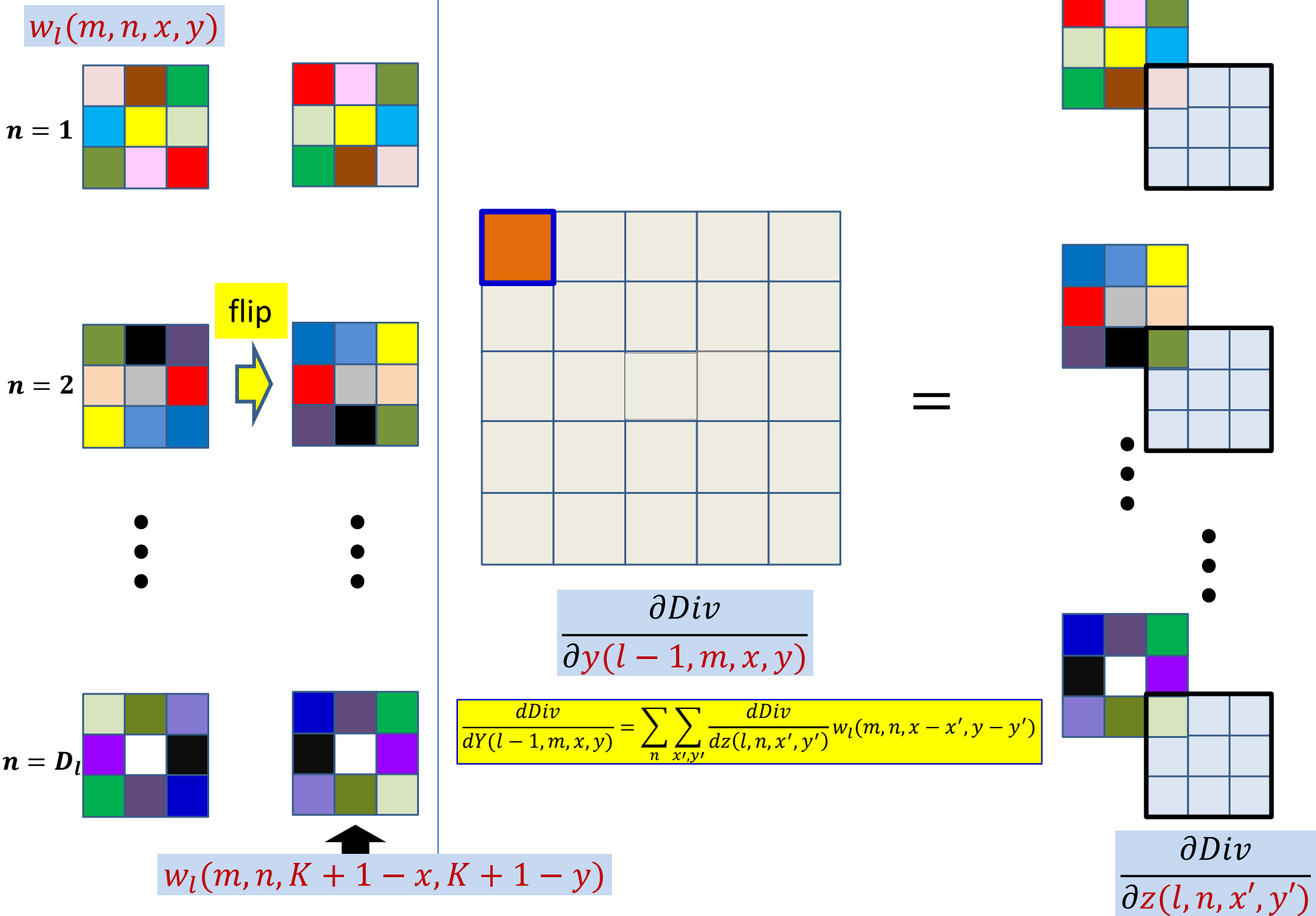
=

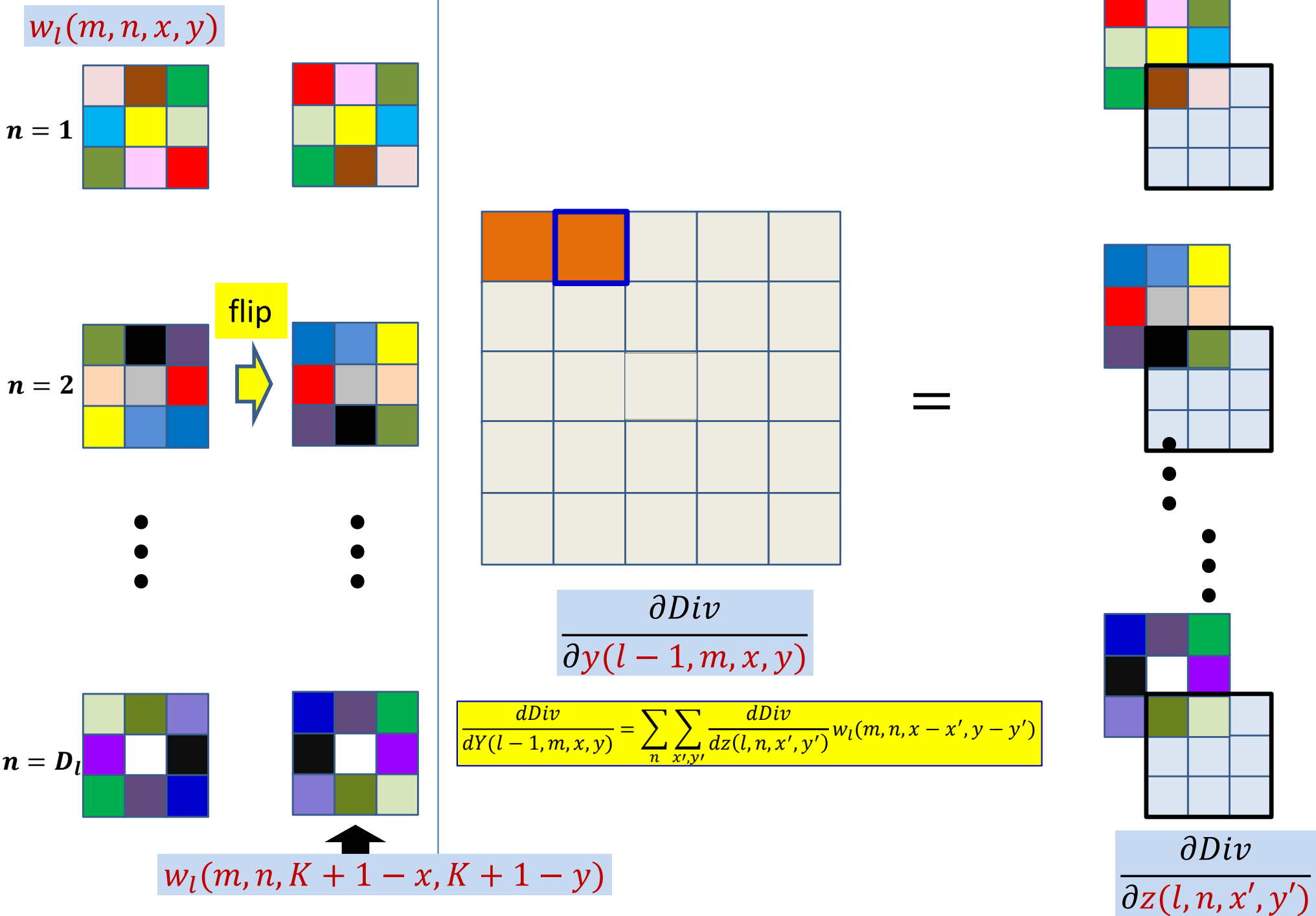
$$\frac{\partial Div}{\partial z(l, n, x', y')}$$

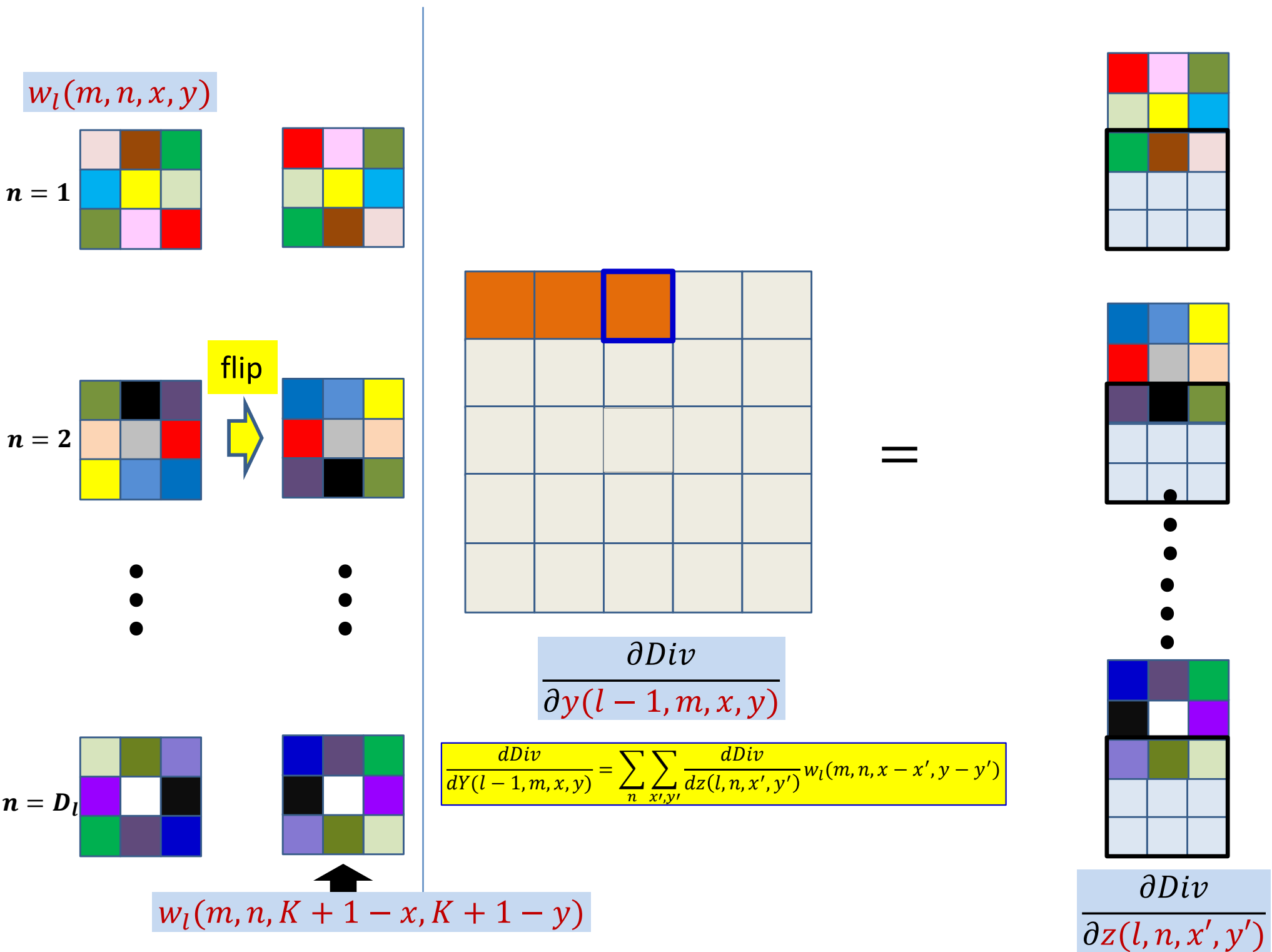


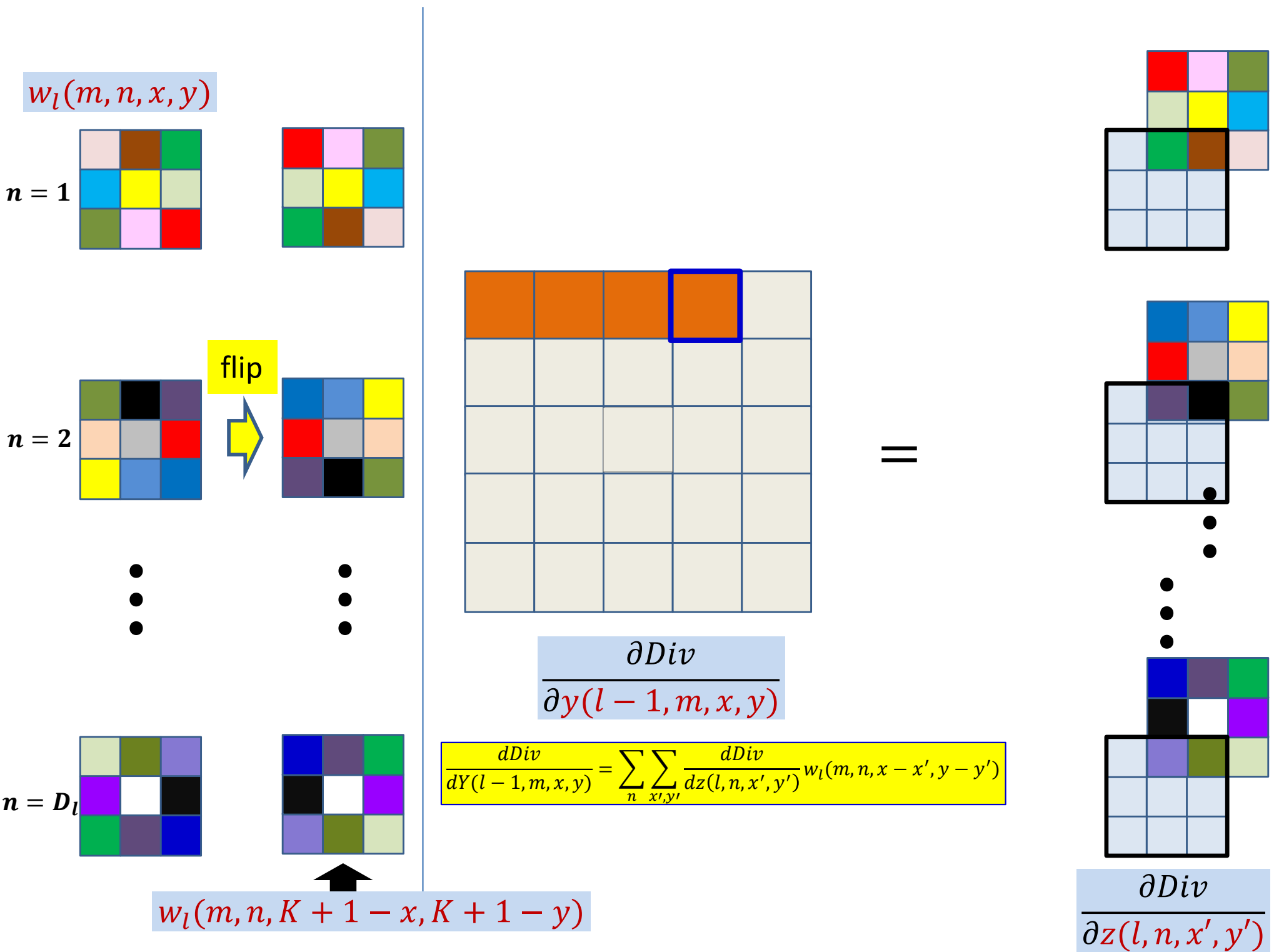
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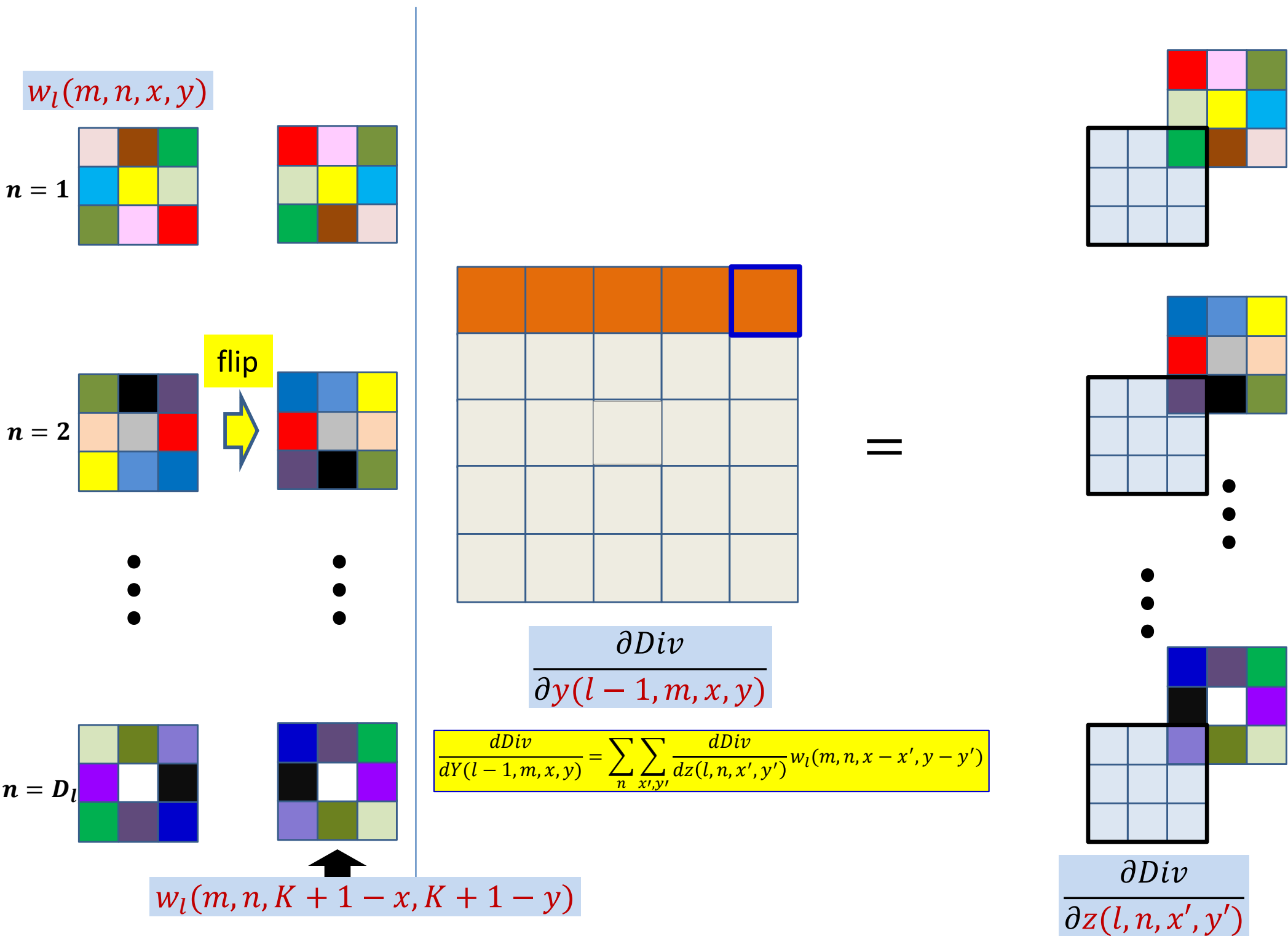


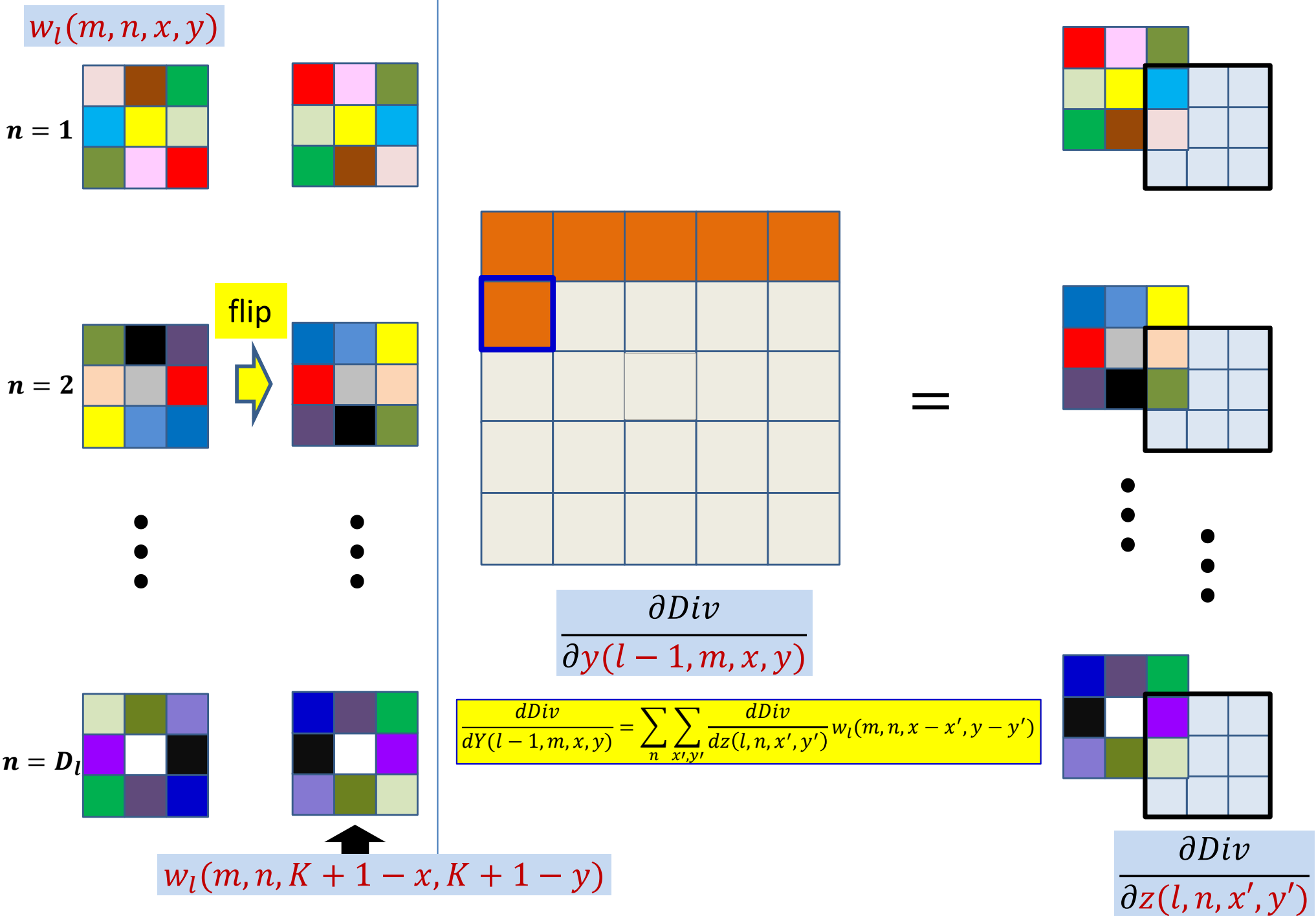


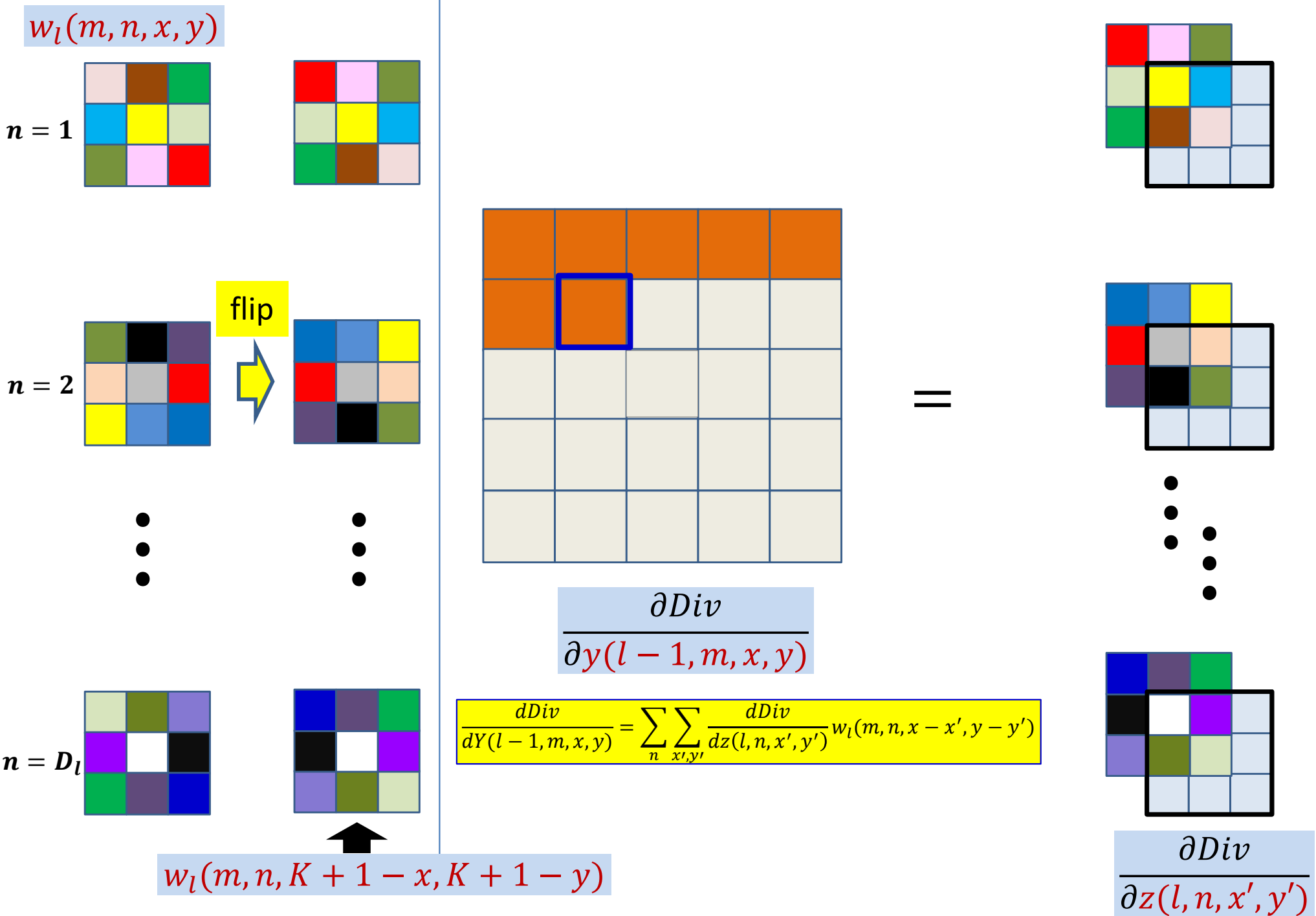


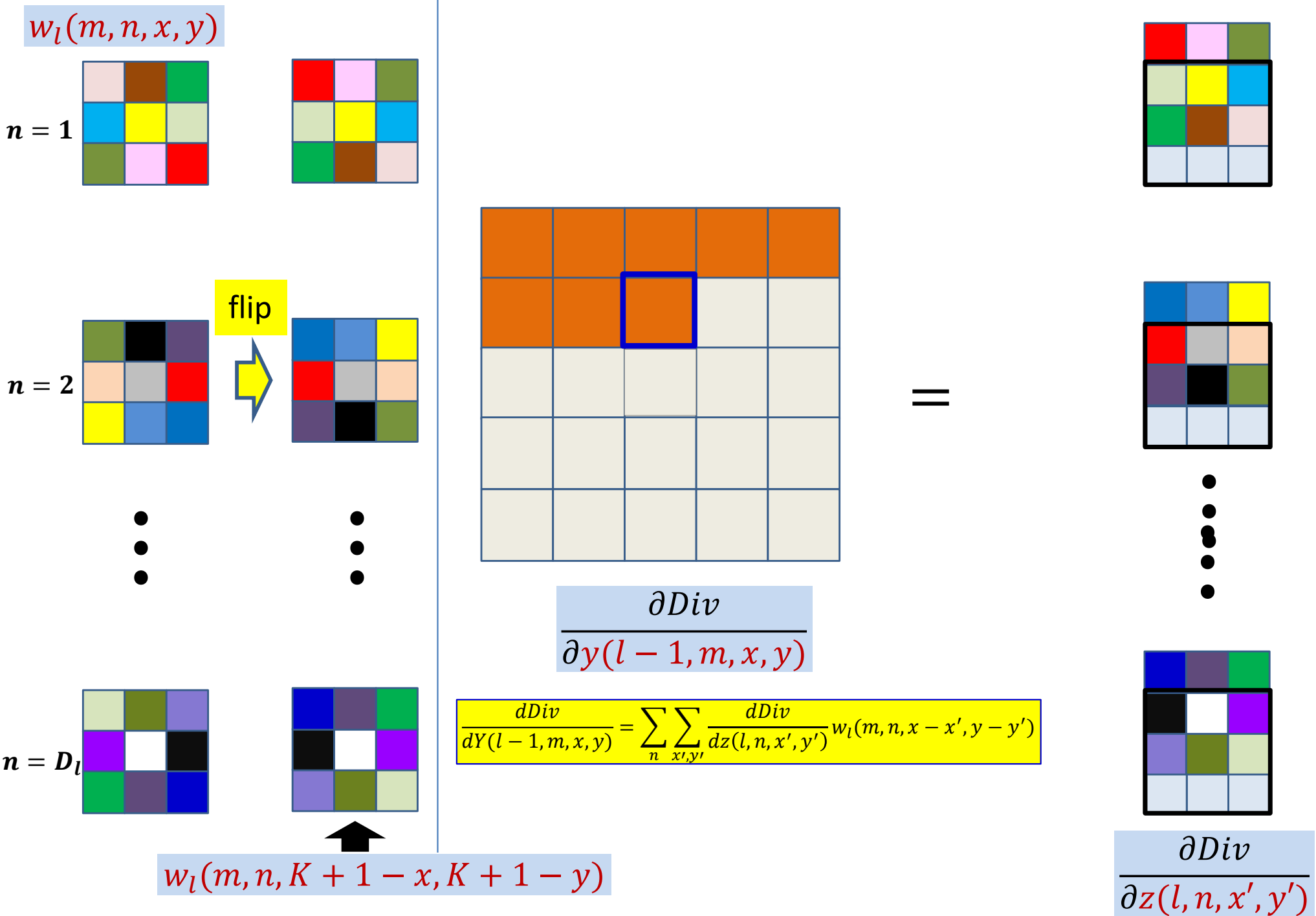


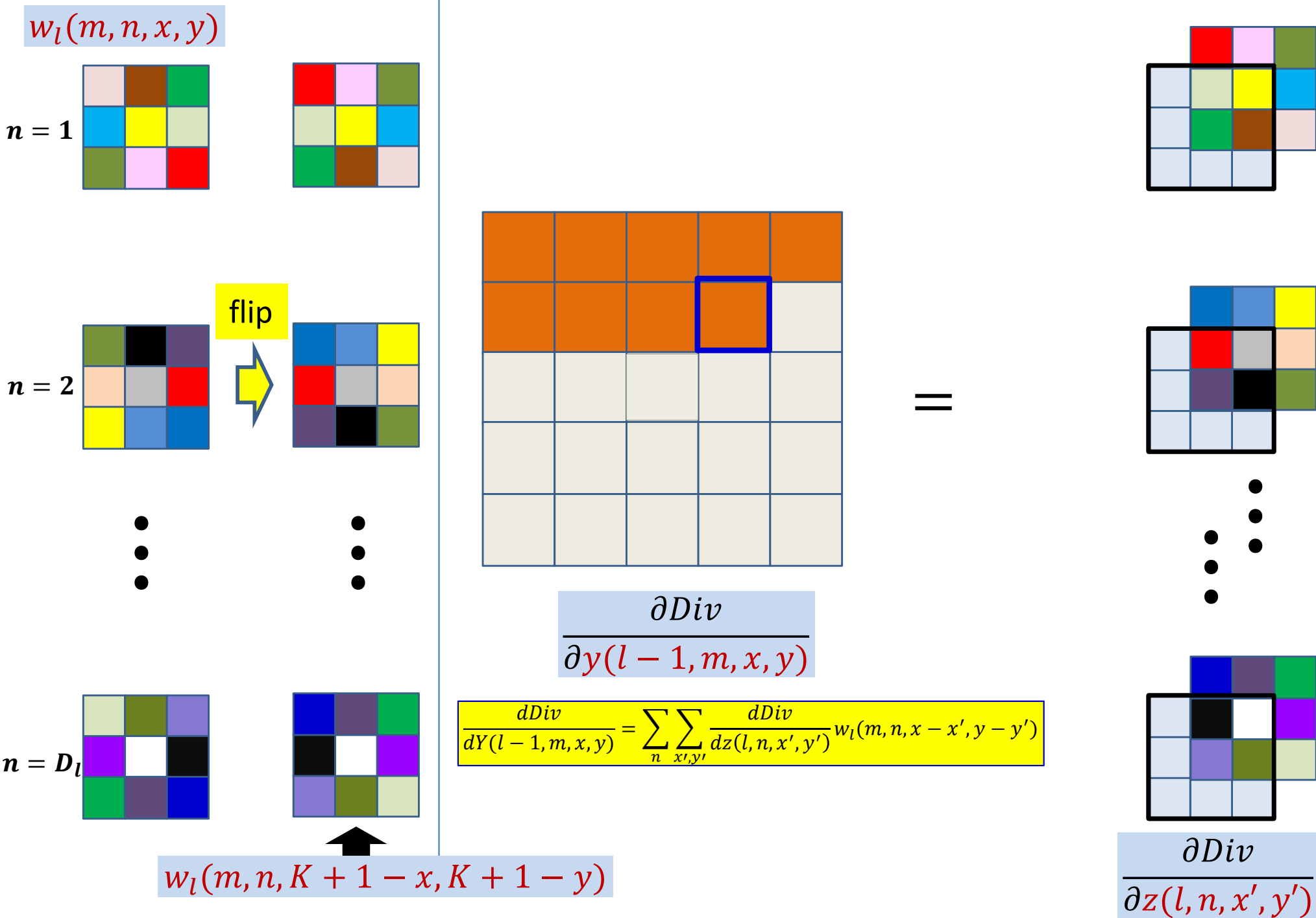


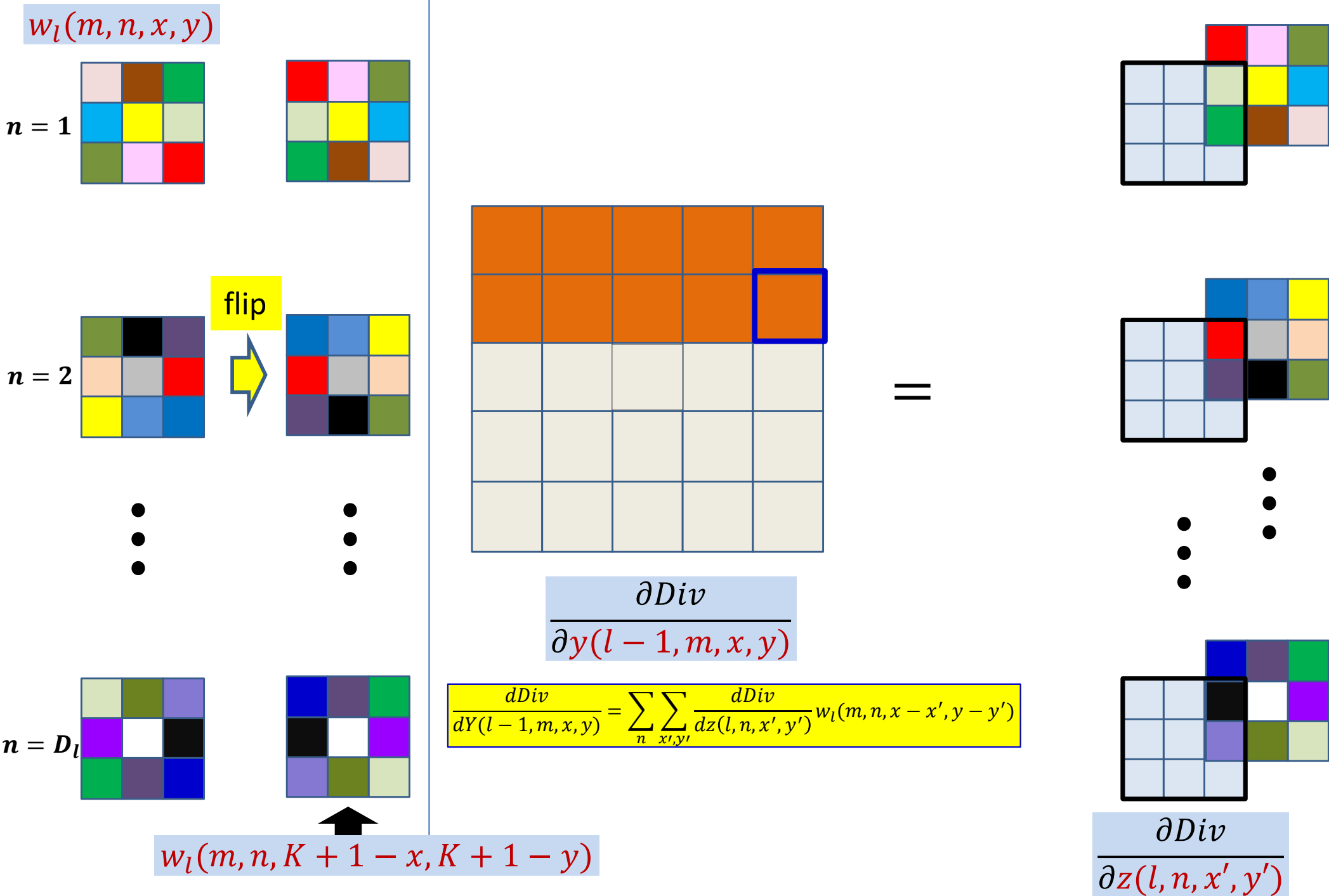






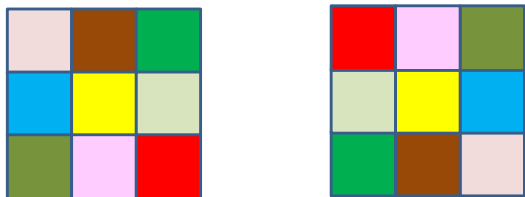




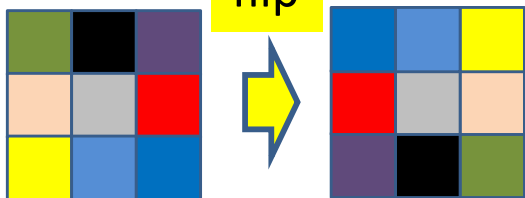


$$w_l(m, n, x, y)$$

$n = 1$



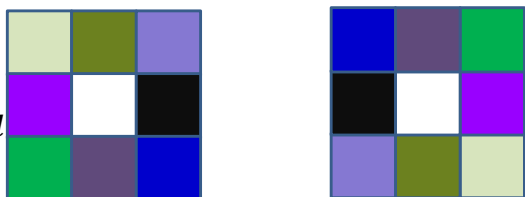
$n = 2$



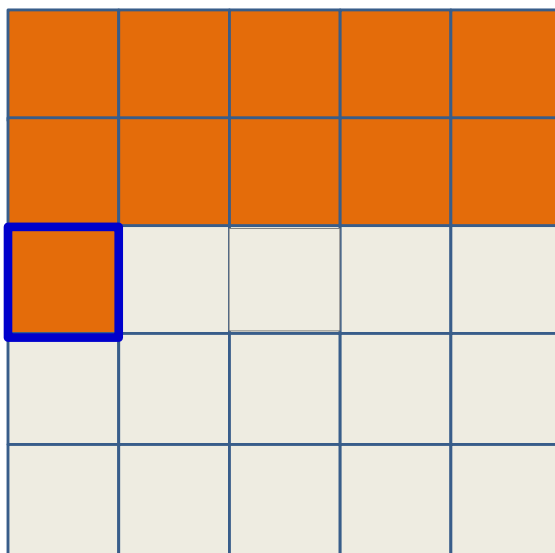
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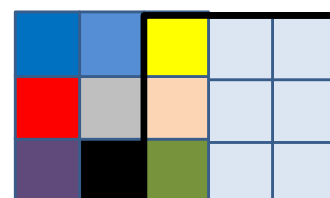
$n = D_l$



$$w_l(m, n, K + 1 - x, K + 1 - y)$$



=



⋮



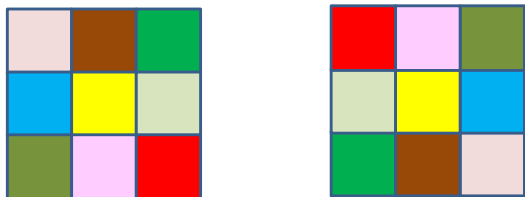
$$\frac{\partial Div}{\partial y(l-1, m, x, y)}$$

$$\frac{dDiv}{dY(l-1, m, x, y)} = \sum_n \sum_{x', y'} \frac{dDiv}{dz(l, n, x', y')} w_l(m, n, x - x', y - y')$$

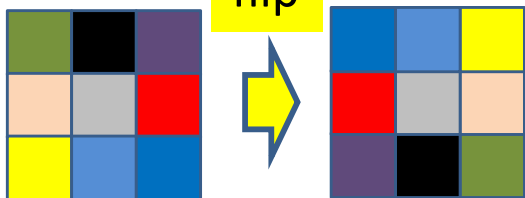
$$\frac{\partial Div}{\partial z(l, n, x', y')}$$

$$w_l(m, n, x, y)$$

$n = 1$



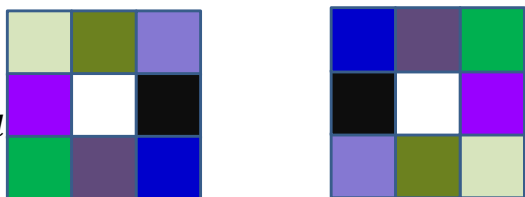
$n = 2$



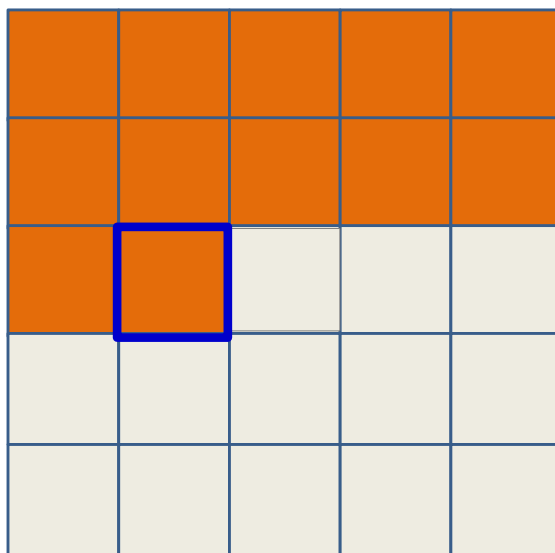
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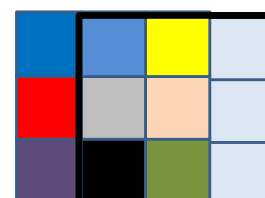
$n = D_l$



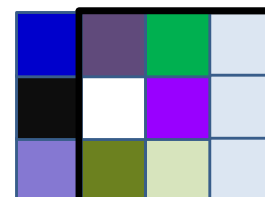
$$w_l(m, n, K + 1 - x, K + 1 - y)$$



=



⋮



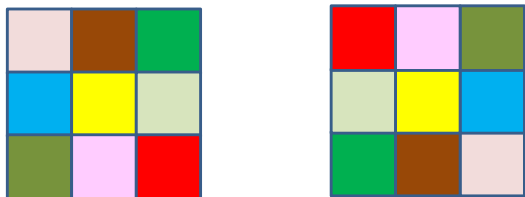
$$\frac{\partial Div}{\partial y(l-1, m, x, y)}$$

$$\frac{dDiv}{dY(l-1, m, x, y)} = \sum_n \sum_{x', y'} \frac{dDiv}{dz(l, n, x', y')} w_l(m, n, x - x', y - y')$$

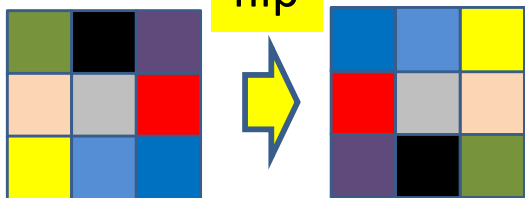
$$\frac{\partial Div}{\partial z(l, n, x', y')}$$

$$w_l(m, n, x, y)$$

$n = 1$



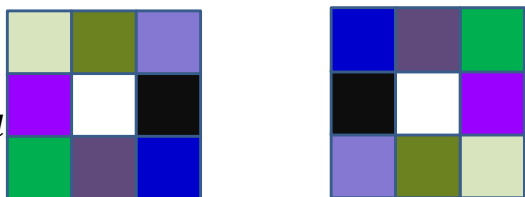
$n = 2$



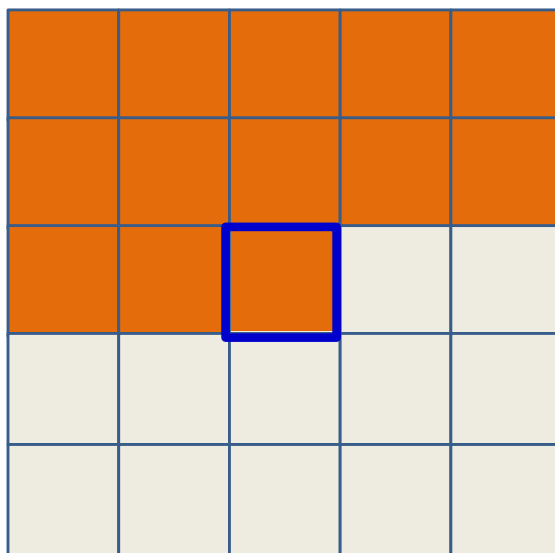
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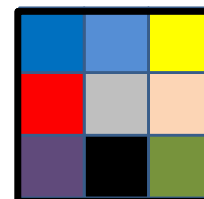
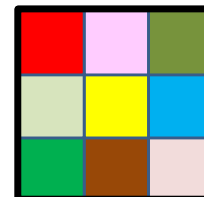
$n = D_l$



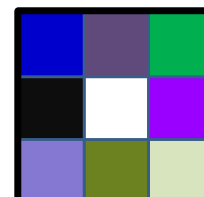
$$w_l(m, n, K + 1 - x, K + 1 - y)$$



$=$



⋮



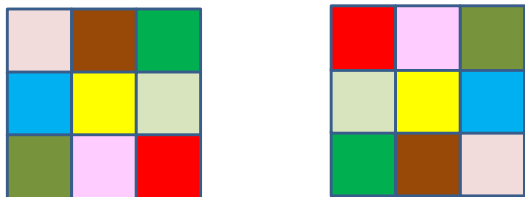
$$\frac{\partial Div}{\partial y(l-1, m, x, y)}$$

$$\frac{dDiv}{dY(l-1, m, x, y)} = \sum_n \sum_{x', y'} \frac{dDiv}{dz(l, n, x', y')} w_l(m, n, x - x', y - y')$$

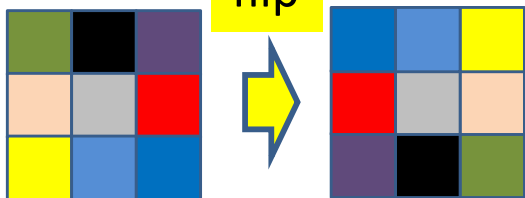
$$\frac{\partial Div}{\partial z(l, n, x', y')}$$

$$w_l(m, n, x, y)$$

$n = 1$



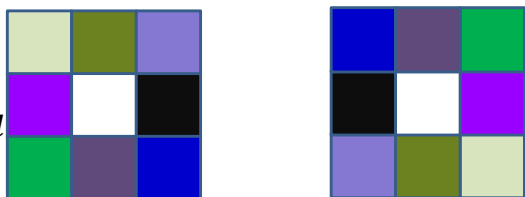
$n = 2$



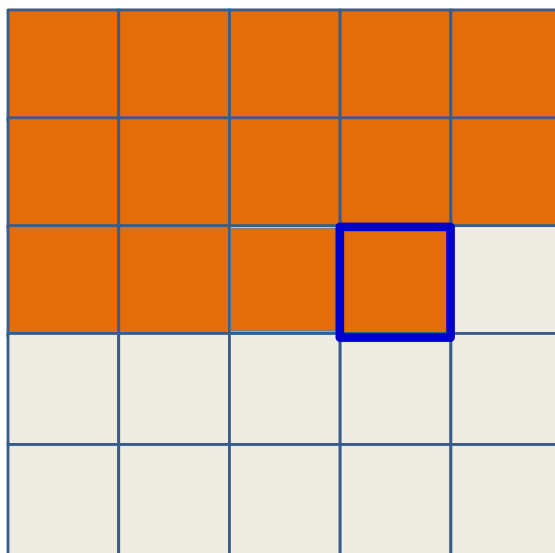
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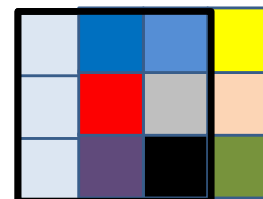
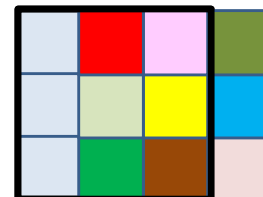
$n = D_l$



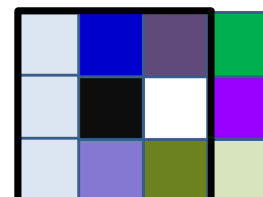
$$w_l(m, n, K + 1 - x, K + 1 - y)$$



=



⋮



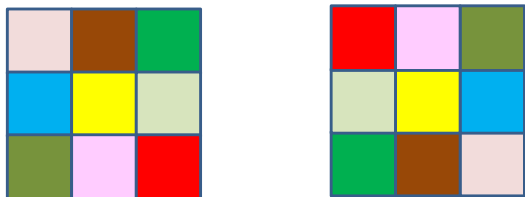
$$\frac{\partial Div}{\partial y(l-1, m, x, y)}$$

$$\frac{dDiv}{dY(l-1, m, x, y)} = \sum_n \sum_{x', y'} \frac{dDiv}{dz(l, n, x', y')} w_l(m, n, x - x', y - y')$$

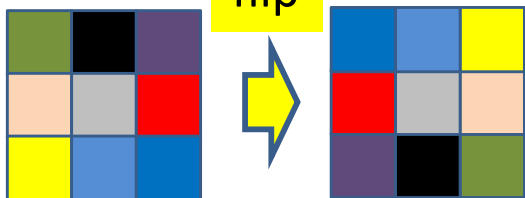
$$\frac{\partial Div}{\partial z(l, n, x', y')}$$

$$w_l(m, n, x, y)$$

$n = 1$



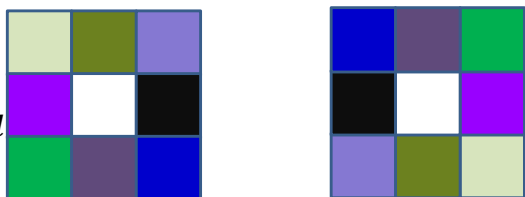
$n = 2$



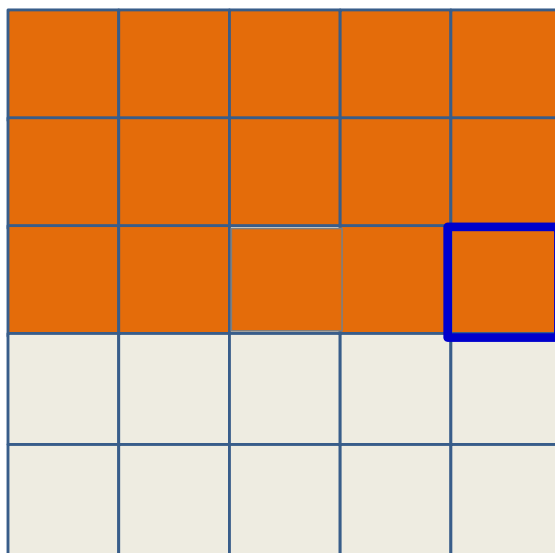
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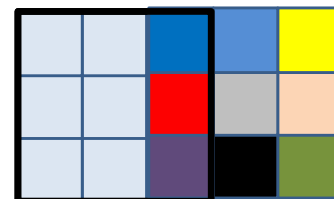
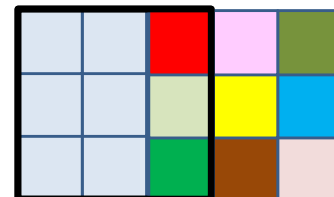
$n = D_l$



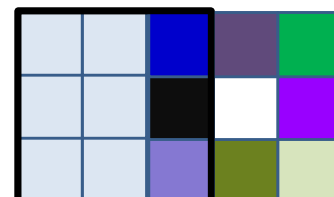
$$w_l(m, n, K + 1 - x, K + 1 - y)$$



=



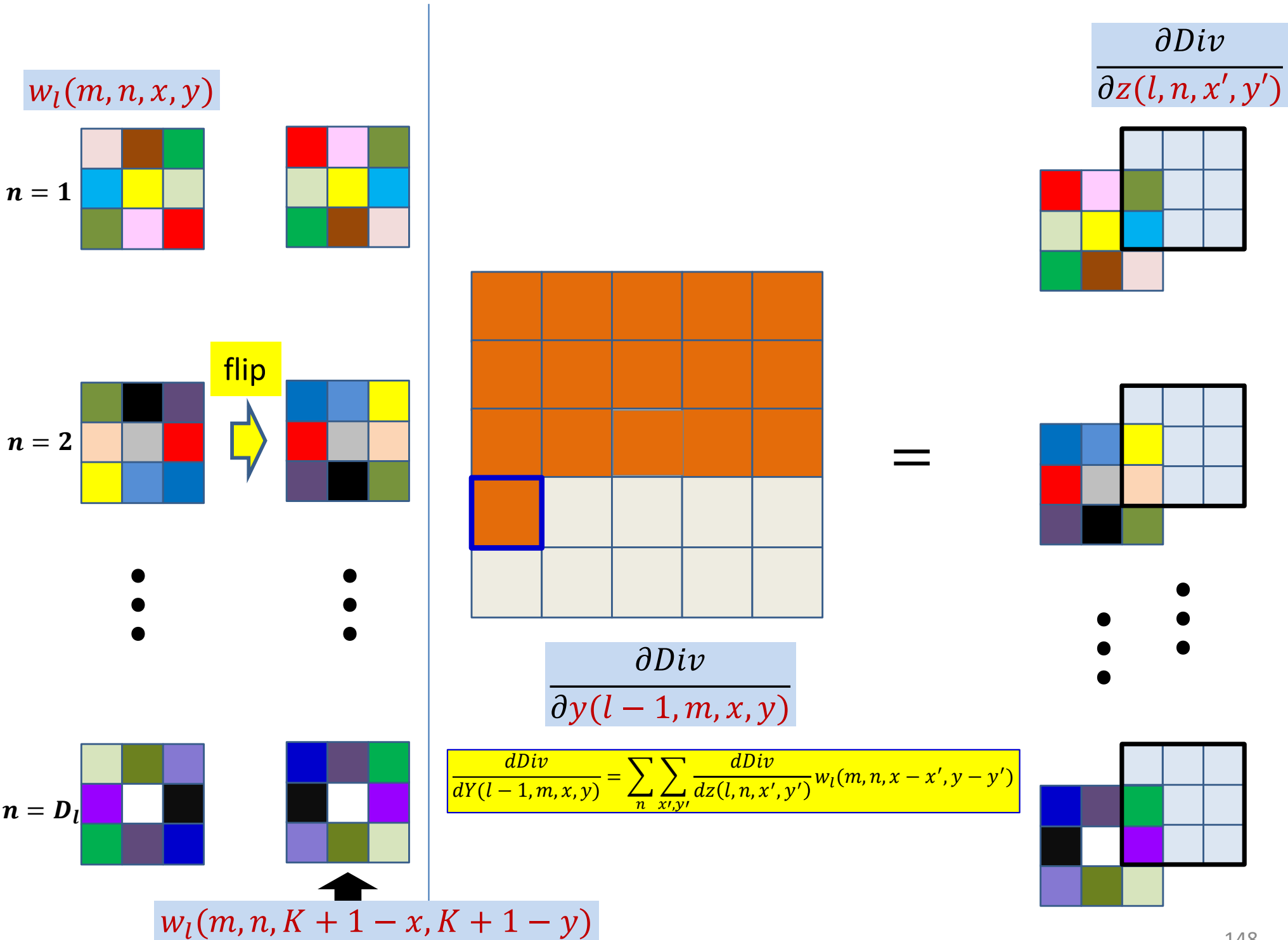
⋮

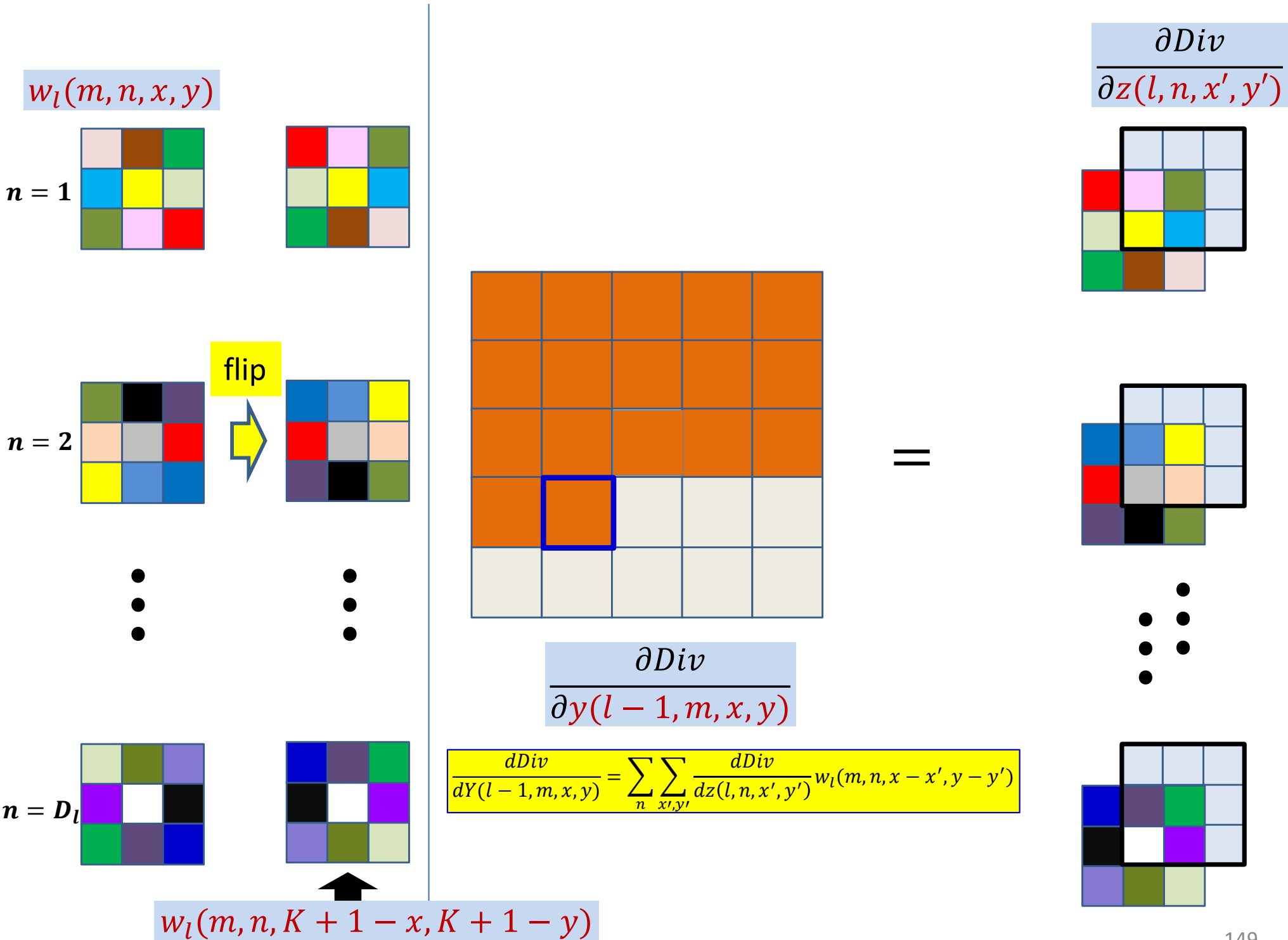


$$\frac{\partial Div}{\partial y(l-1, m, x, y)}$$

$$\frac{dDiv}{dY(l-1, m, x, y)} = \sum_n \sum_{x', y'} \frac{dDiv}{dz(l, n, x', y')} w_l(m, n, x - x', y - y')$$

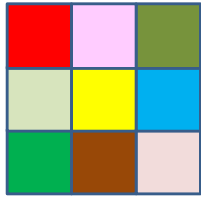
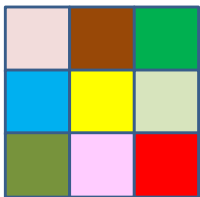
$$\frac{\partial Div}{\partial z(l, n, x', y')}$$



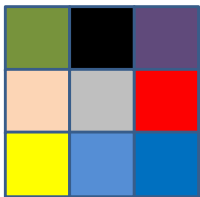


$$w_l(m, n, x, y)$$

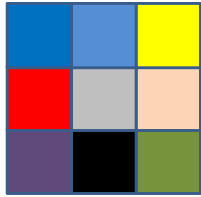
$n = 1$



$n = 2$



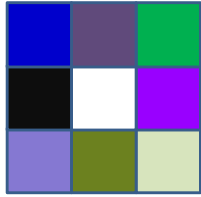
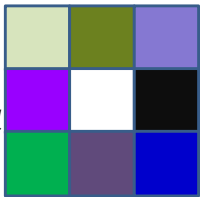
flip



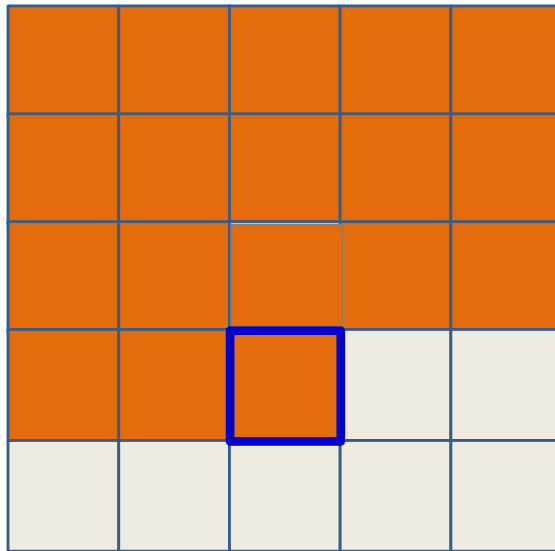
⋮

⋮

$n = D_l$



$$w_l(m, n, K + 1 - x, K + 1 - y)$$

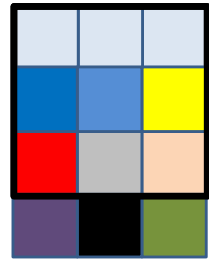
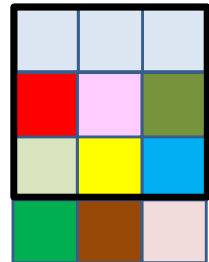


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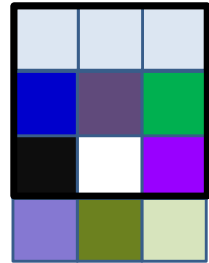
$$\frac{\partial Div}{\partial y(l-1, m, x, y)}$$

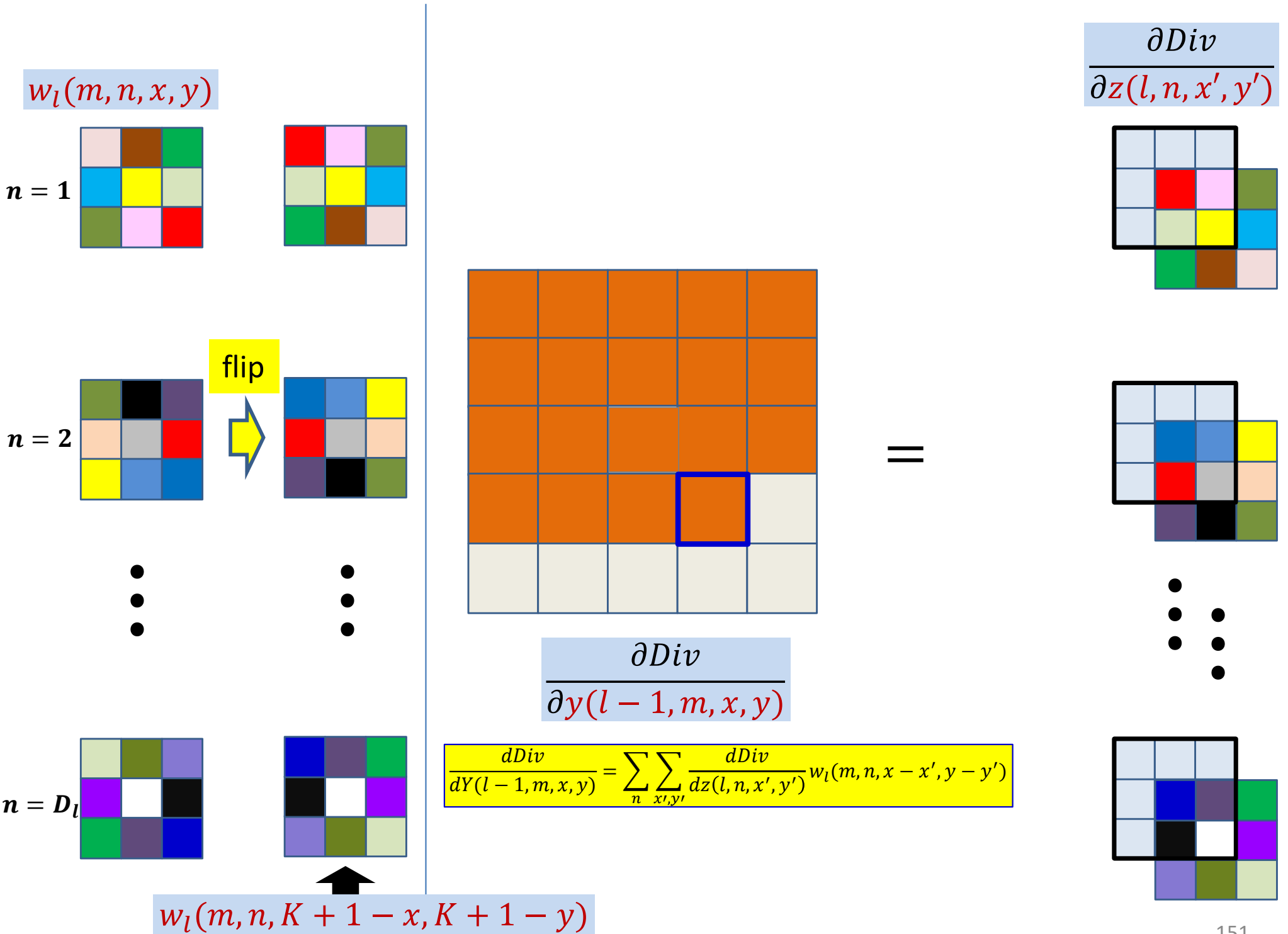
$$\frac{dDiv}{dY(l-1, m, x, y)} = \sum_n \sum_{x', y'} \frac{dDiv}{dz(l, n, x', y')} w_l(m, n, x - x', y - y')$$

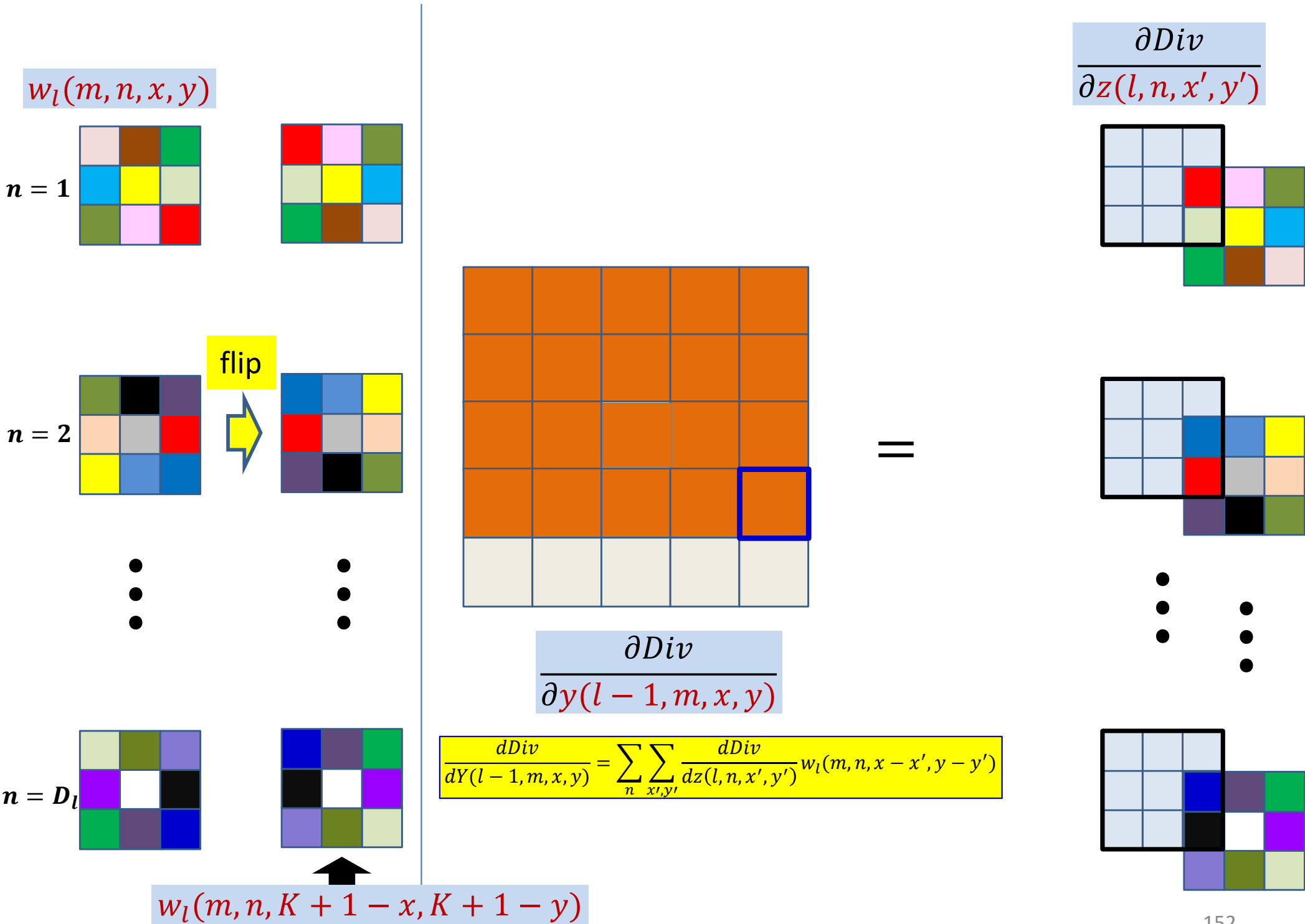
$$\frac{\partial Div}{\partial z(l, n, x', y')}$$

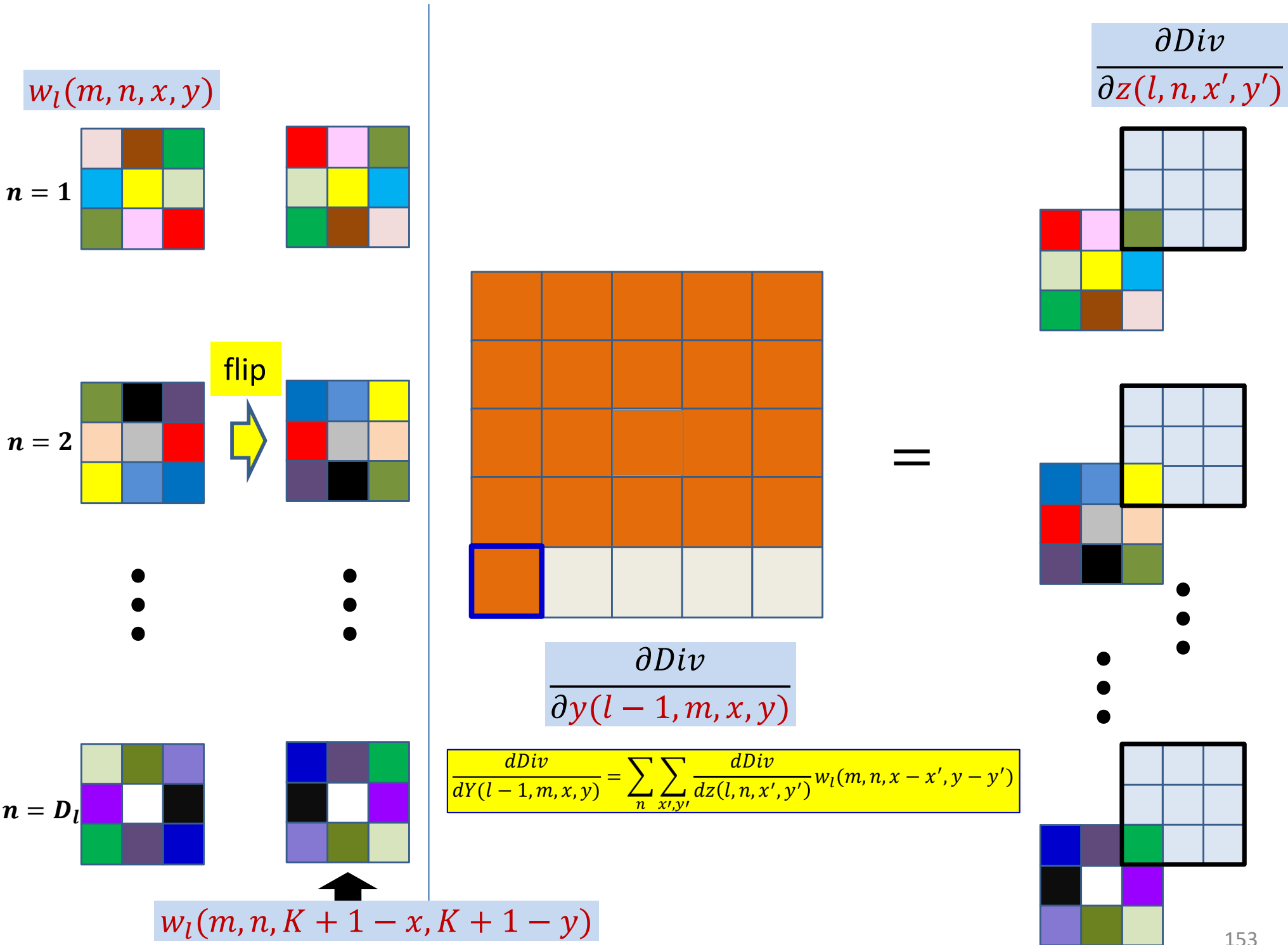


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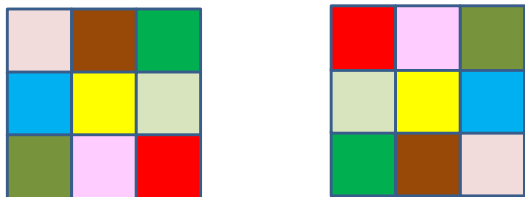




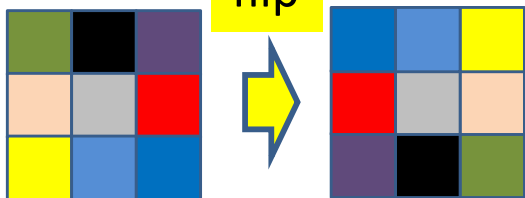


$$w_l(m, n, x, y)$$

$n = 1$



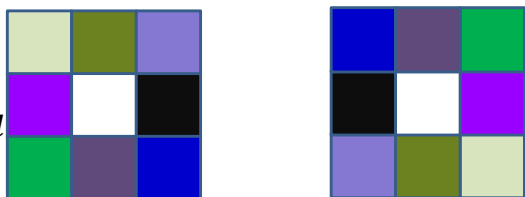
$n = 2$



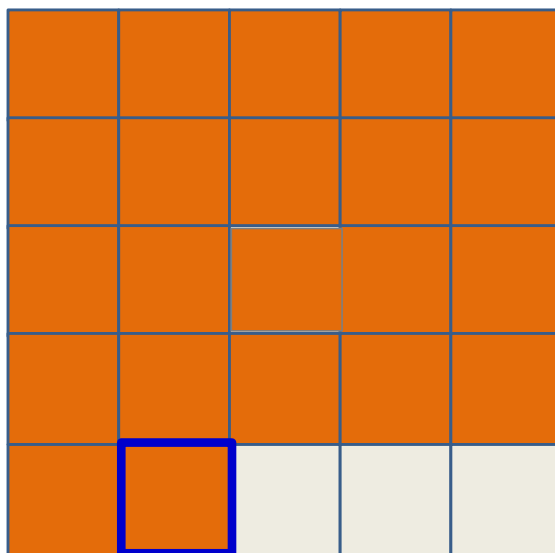
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$n = D_l$



$$w_l(m, n, K + 1 - x, K + 1 - y)$$

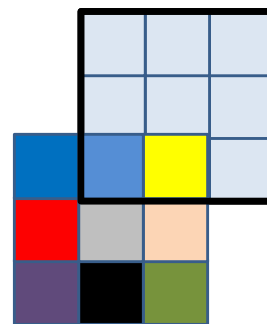
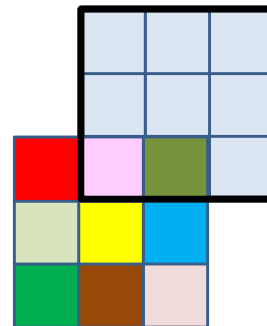


$$\frac{\partial Div}{\partial y(l-1, m, x, y)}$$

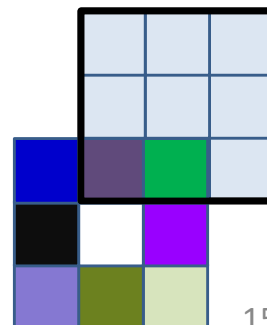
$$\frac{dDiv}{dY(l-1, m, x, y)} = \sum_n \sum_{x', y'} \frac{dDiv}{dz(l, n, x', y')} w_l(m, n, x - x', y - y')$$

=

$$\frac{\partial Div}{\partial z(l, n, x', y')}$$

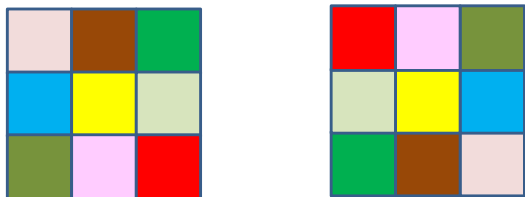


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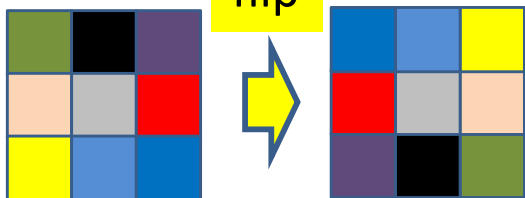


$$w_l(m, n, x, y)$$

$n = 1$



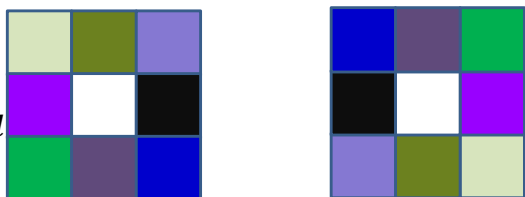
$n = 2$



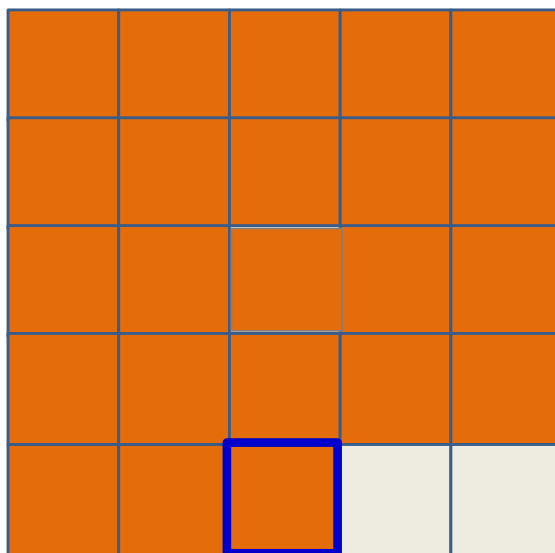
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⋮

$n = D_l$



$$w_l(m, n, K + 1 - x, K + 1 - y)$$

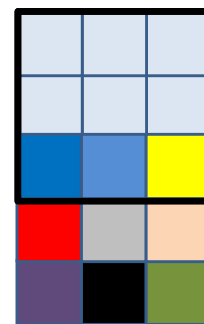
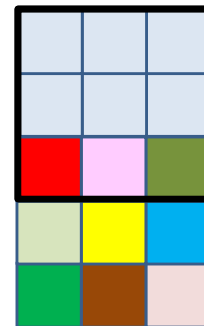


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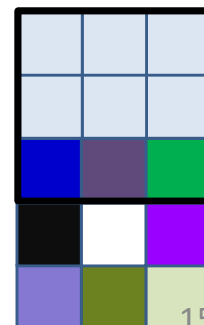
$$\frac{\partial Div}{\partial y(l-1, m, x, y)}$$

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$$\frac{\partial Div}{\partial z(l, n, x', y')}$$

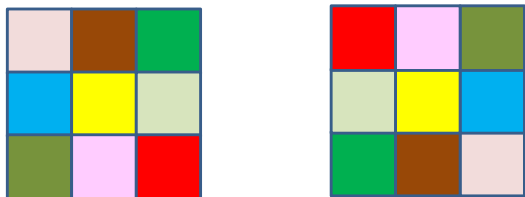


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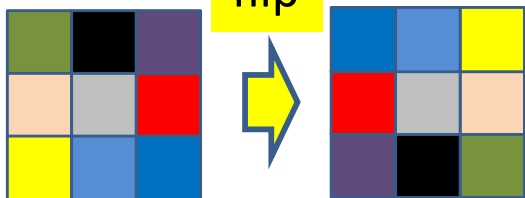


$$w_l(m, n, x, y)$$

$n = 1$



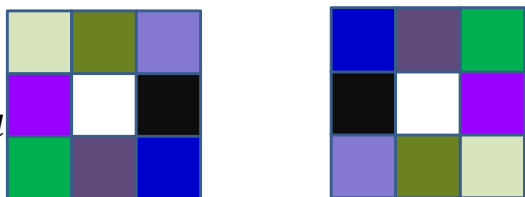
$n = 2$



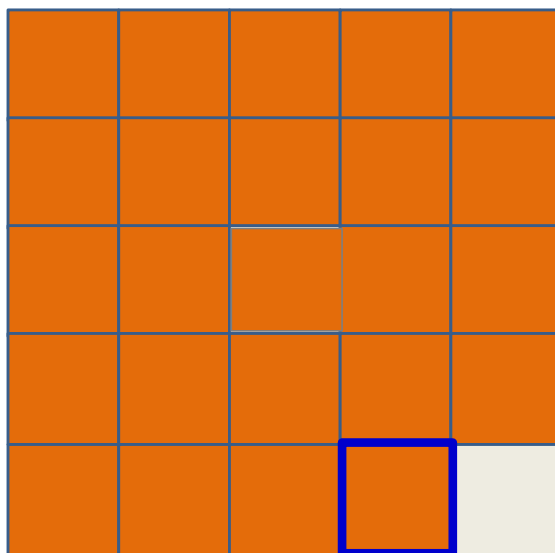
⋮

⋮

$n = D_l$



$$w_l(m, n, K + 1 - x, K + 1 - y)$$

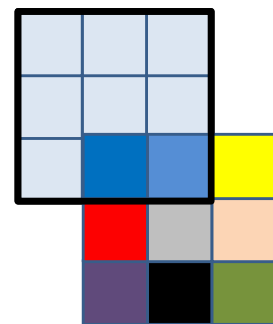
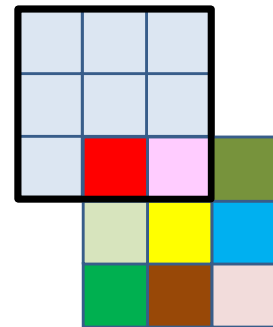


$$\frac{\partial Div}{\partial y(l-1, m, x, y)}$$

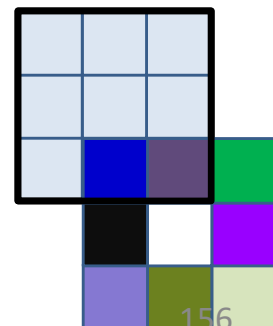
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$$\frac{dDiv}{dY(l-1, m, x, y)} = \sum_n \sum_{x', y'} \frac{dDiv}{dz(l, n, x', y')} w_l(m, n, x - x', y - y')$$

$$\frac{\partial Div}{\partial z(l, n, x', y')}$$

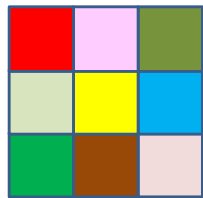
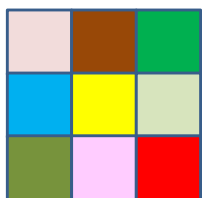


⋮

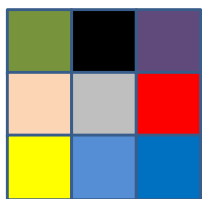


$$w_l(m, n, x, y)$$

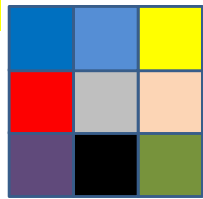
$n = 1$



$n = 2$



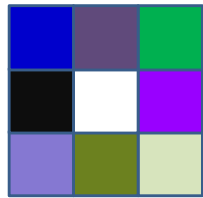
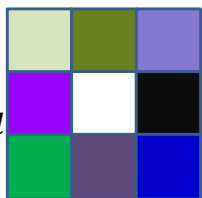
flip



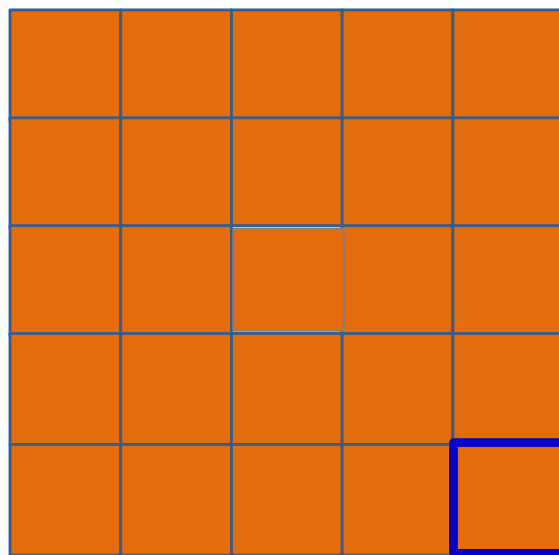
⋮

⋮

$n = D_l$



$$w_l(m, n, K + 1 - x, K + 1 - y)$$

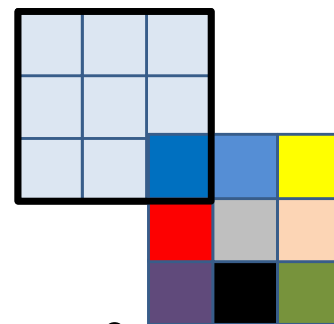
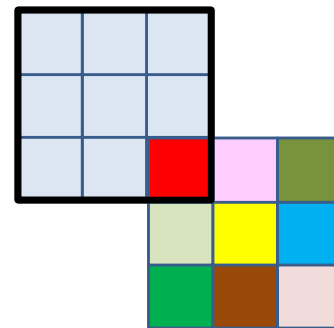


=

$$\frac{\partial Div}{\partial y(l-1, m, x, y)}$$

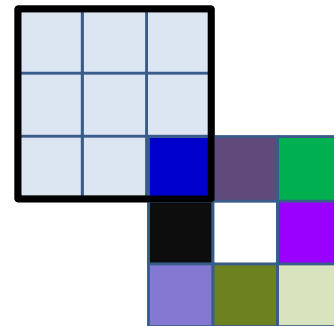
$$\frac{dDiv}{dY(l-1, m, x, y)} = \sum_n \sum_{x', y'} \frac{dDiv}{dz(l, n, x', y')} w_l(m, n, x - x', y - y')$$

$$\frac{\partial Div}{\partial z(l, n, x', y')}$$

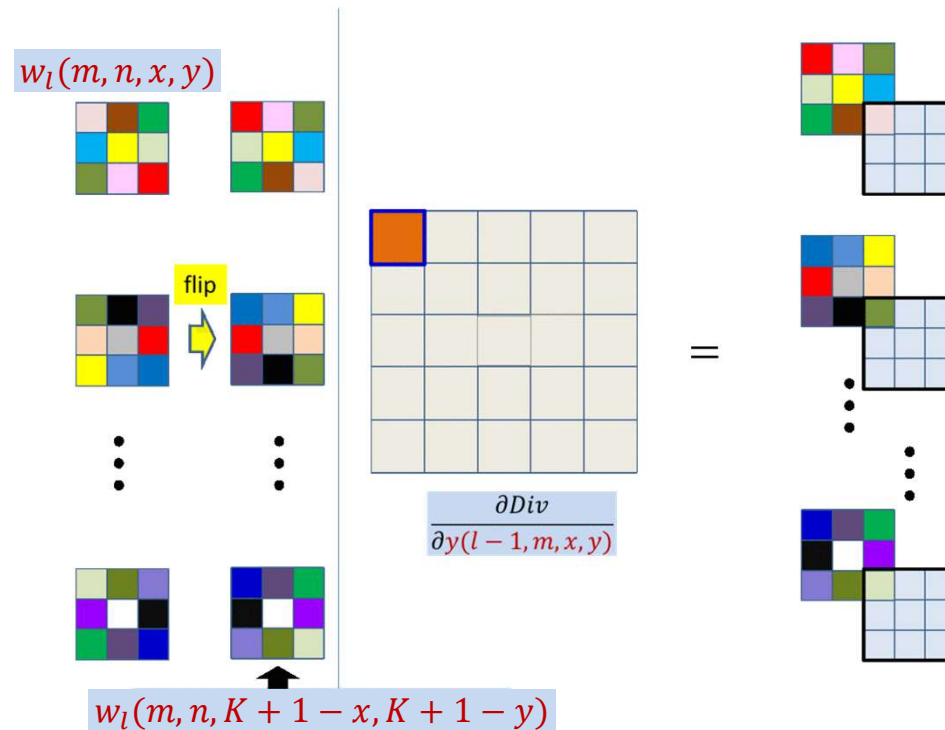


⋮

⋮

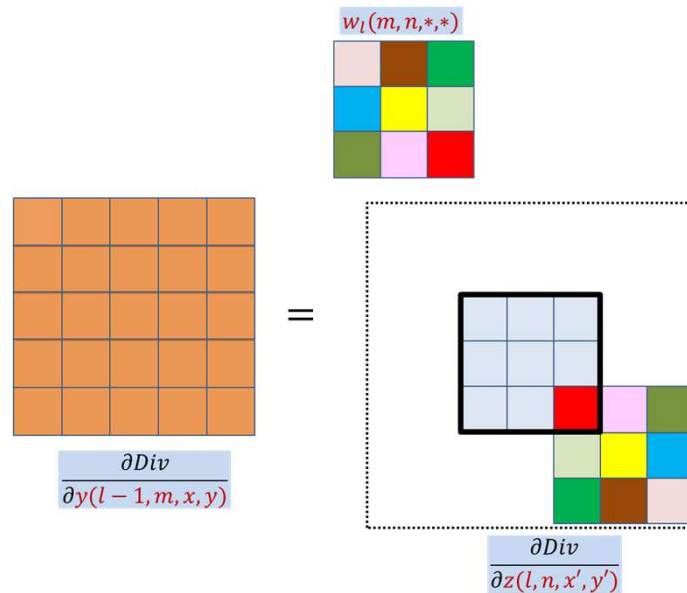


Computing the derivative for $Y(l - 1, m)$



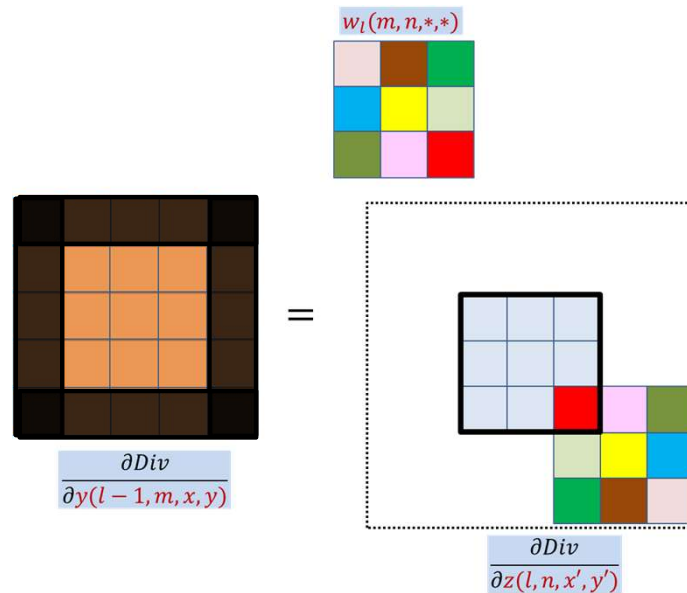
- This is just a convolution of the zero-padded $\frac{\partial Div}{\partial z(l, n, x, y)}$ maps by the transposed and flipped filter
 - After zero padding it first with $K - 1$ zeros on every side

The size of the Y-derivative map



- We continue to compute elements for the derivative Y map as long as the (flipped) filter has at least one element in the (unpadded) derivative Zmap
 - I.e. so long as the Y derivative is non-zero
- The size of the Y derivative map will be $(H + K - 1) \times (W + K - 1)$
 - H and W are the height and width of the Zmap
- This will be the size of the actual Y map that was originally convolved

The size of the Y-derivative map



- If the Y map was zero-padded in the forward pass, the derivative map will be the size of the *zero-padded* map
 - The zero padding regions must be deleted before further backprop

Poll 3: @760

Select all statements that are true about how to compute the derivative of the divergence w.r.t l th layer activation maps by backpropagation

- To compute the derivative w.r.t. the m th activation map of the l th convolutional layer, we must select the m th “planes” of all the $(l+1)$ th layer filters
- The selected filter planes must be flipped left-right and up-down
- They must convolve the derivative (maps) for the $(l+1)$ th layer affine values
- The output of the convolution must be flipped back left-right and up-down

Poll 3

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Overall algorithm for computing derivatives w.r.t. $Y(l - 1)$

- Given the derivatives $\frac{dDiv}{dz(l, n, x, y)}$
- Compute derivatives using:

$$\frac{dDiv}{dY(l - 1, m, x, y)} = \sum_n \sum_{x', y'} \frac{dDiv}{dz(l, n, x', y')} w_l(m, n, x - x', y - y')$$

Can be computed by convolution with flipped filter

l is layer index

Derivatives for a single layer l :

Vector notation

The weight $W(l,m)$ is a 3D $D_{l-1} \times K_l \times K_l$

Assuming dz has already been obtained via backprop

```
dzpad = zeros(Dl × (Hl + 2(Kl - 1)) × (Wl + 2(Kl - 1))) # zeropad
for j = 1:Dl
    for i = 1:Dl-1    # Transpose and flip
        Wflip(i,j, :, :) = flipLeftRight(flipUpDown(W(l,i,j, :, :)))
        dzpad(j, Kl:Kl+Hl-1, Kl:Kl+Wl-1) = dz(l,j, :, :) #center map
    end
end
```

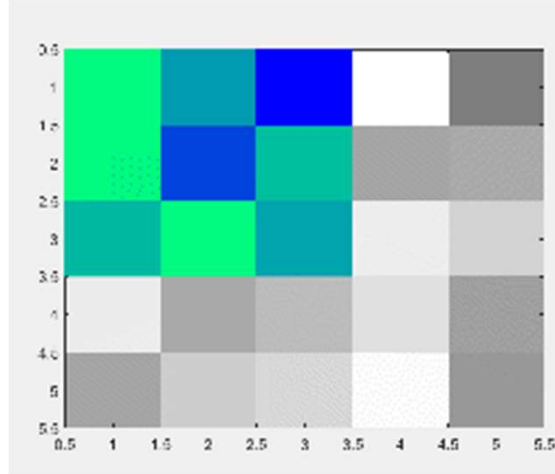
```
for j = 1:Dl-1
    for x = 1:Wl-1
        for y = 1:Hl-1
            segment = dzpad(:, x:x+Kl-1, y:y+Kl-1) #3D tensor
            dy(l-1, j, x, y) = Wflip.segment #tensor inner prod.
```

Backpropagating through affine map

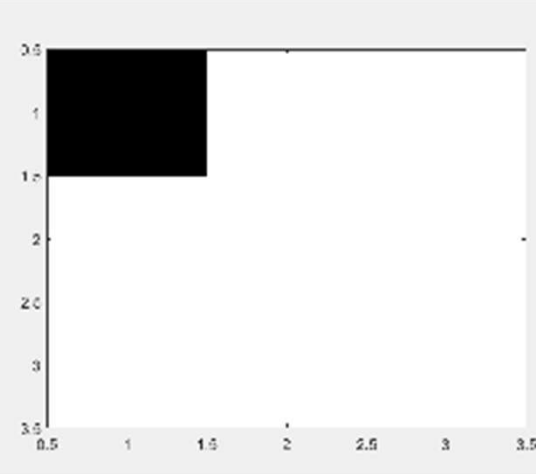
- Forward affine computation:
 - Compute affine maps $z(l, n, x, y)$ from previous layer maps $y(l - 1, m, x, y)$ and filters $w_l(m, n, x, y)$
- Backpropagation: Given $\frac{dDiv}{dz(l, n, x, y)}$
 - ✓ Compute derivative w.r.t. $y(l - 1, m, x, y)$
 - Compute derivative w.r.t. $w_l(m, n, x, y)$

The derivatives for the filters

$Y(l-1, m) \otimes w_l(m, n)$



$Z(l, n)$

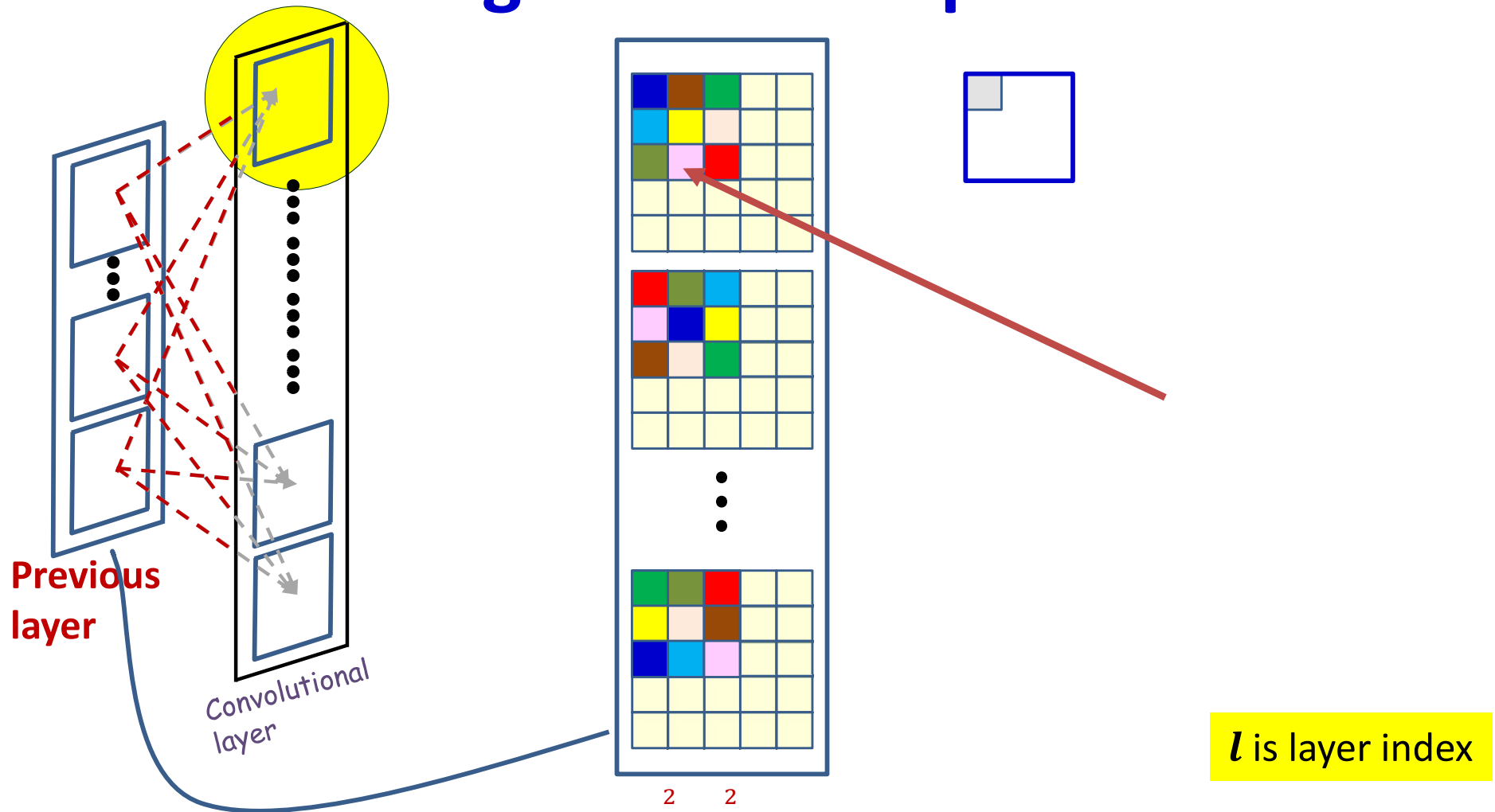


l is layer index

$$z(l, n, x, y) = \sum_m \sum_{x', y'} w_l(m, n, x', y') y(l-1, m, x+x', y+y') + b_l(n)$$

- Each **filter component** $w_l(m, n, x', y')$ affects several $z(l, n, x, y)$ but only within a *single* output affine ($z(l, n, *, *)$) map/channel
 - And is also linked to several $y(l-1, m, *, *)$ but only within a single input channel
 - A single filter channel $w_l(m, n, *, *)$ connects $y(l-1, m, *, *)$ to $z(l, n, *, *)$
 - Consider the contribution of one filter component: $w_l(m, n, i, j)$ (e.g. $w_l(m, n, 1, 2)$) in the above animation for illustration

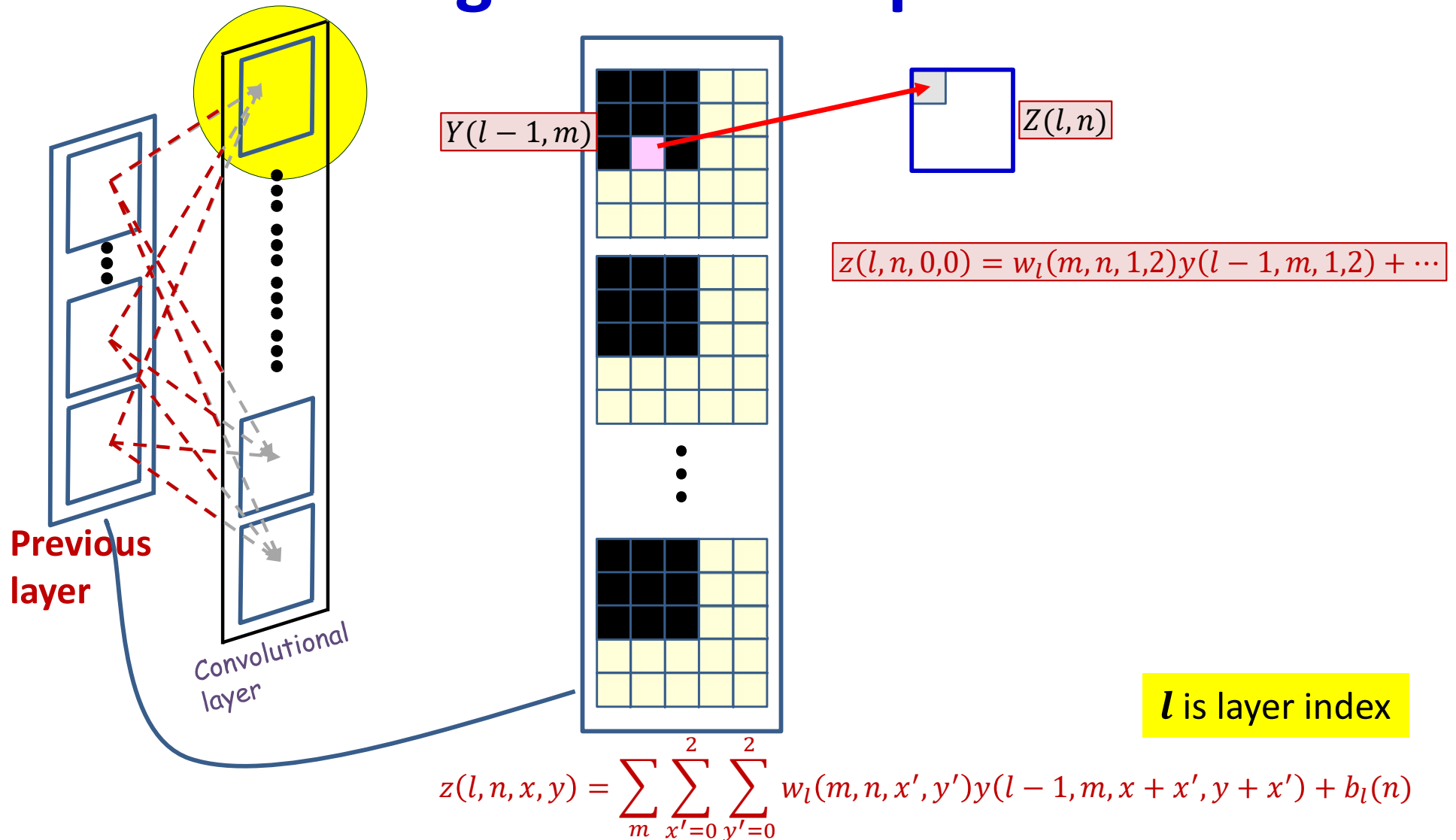
Convolution: the contribution of a single filter component



$$z(l, n, x, y) = \sum_m \sum_{x'=0}^2 \sum_{y'=0}^2 w_l(m, n, x', y') y(l-1, m, x+x', y+y') + b_l(n)$$

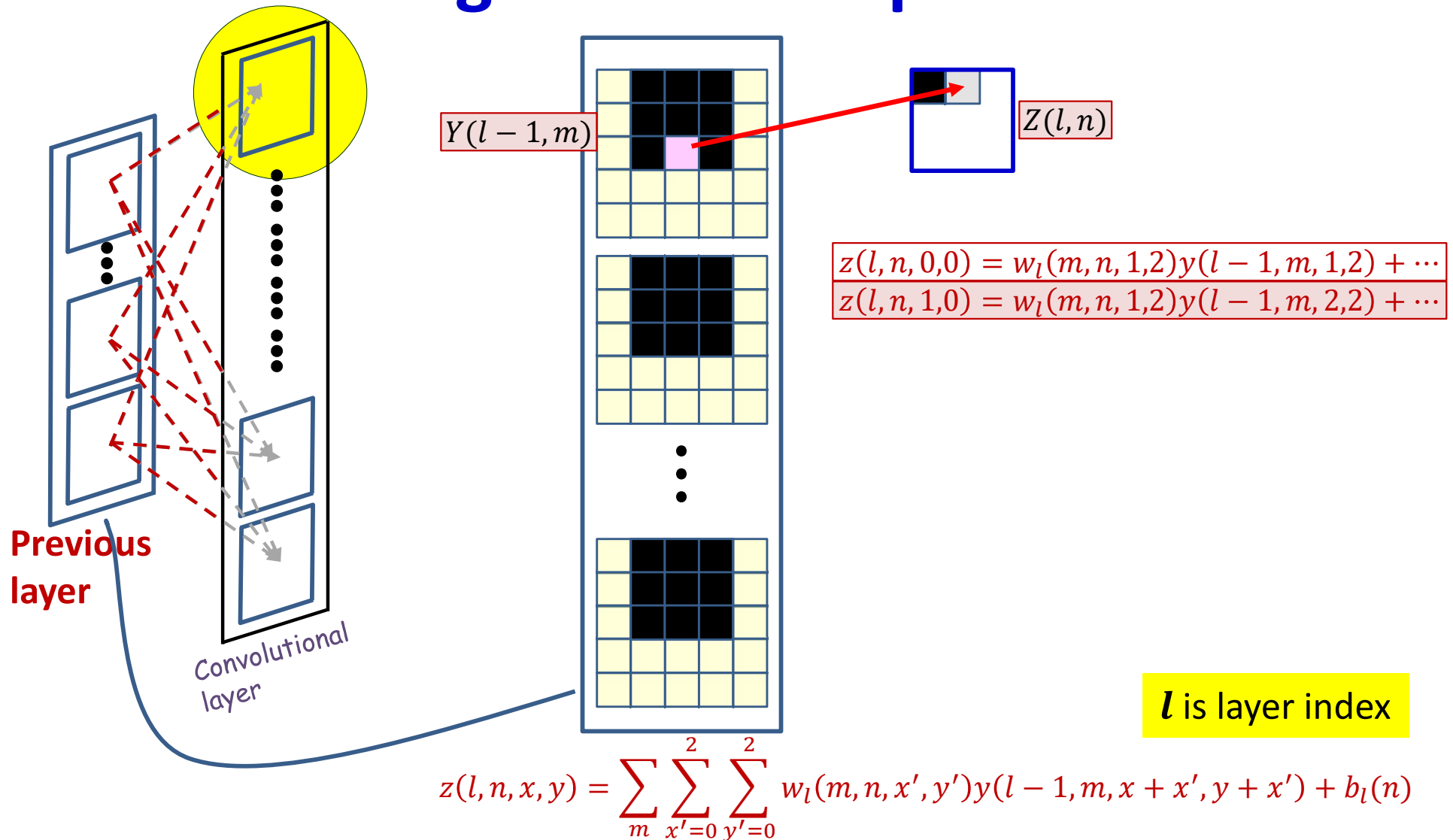
- Each **filter component** $w_l(m, n, i, j)$ affects several $z(l, n, x, y)$ within the n th output affine map

Convolution: the contribution of a single filter component



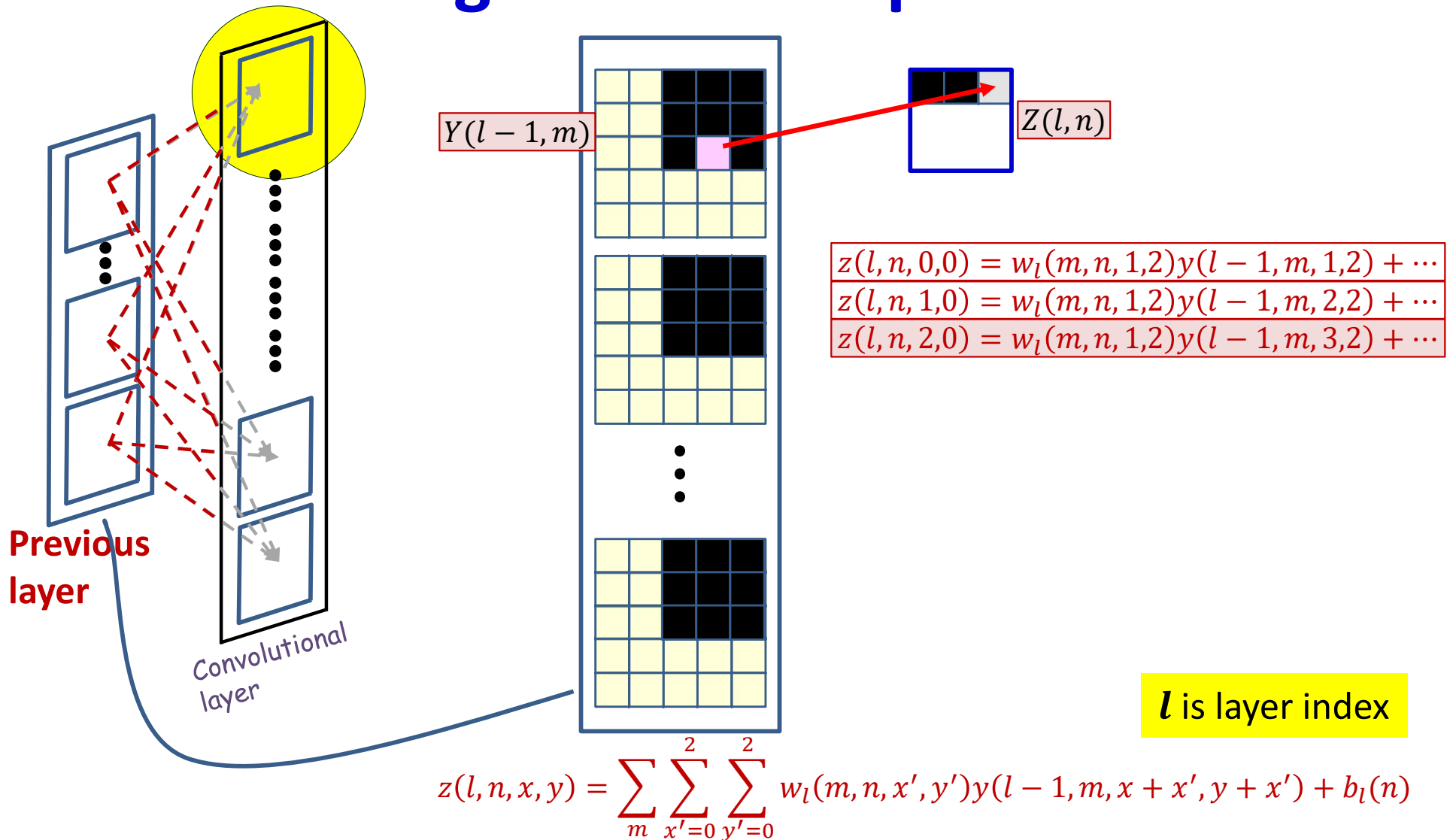
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Convolution: the contribution of a single filter component



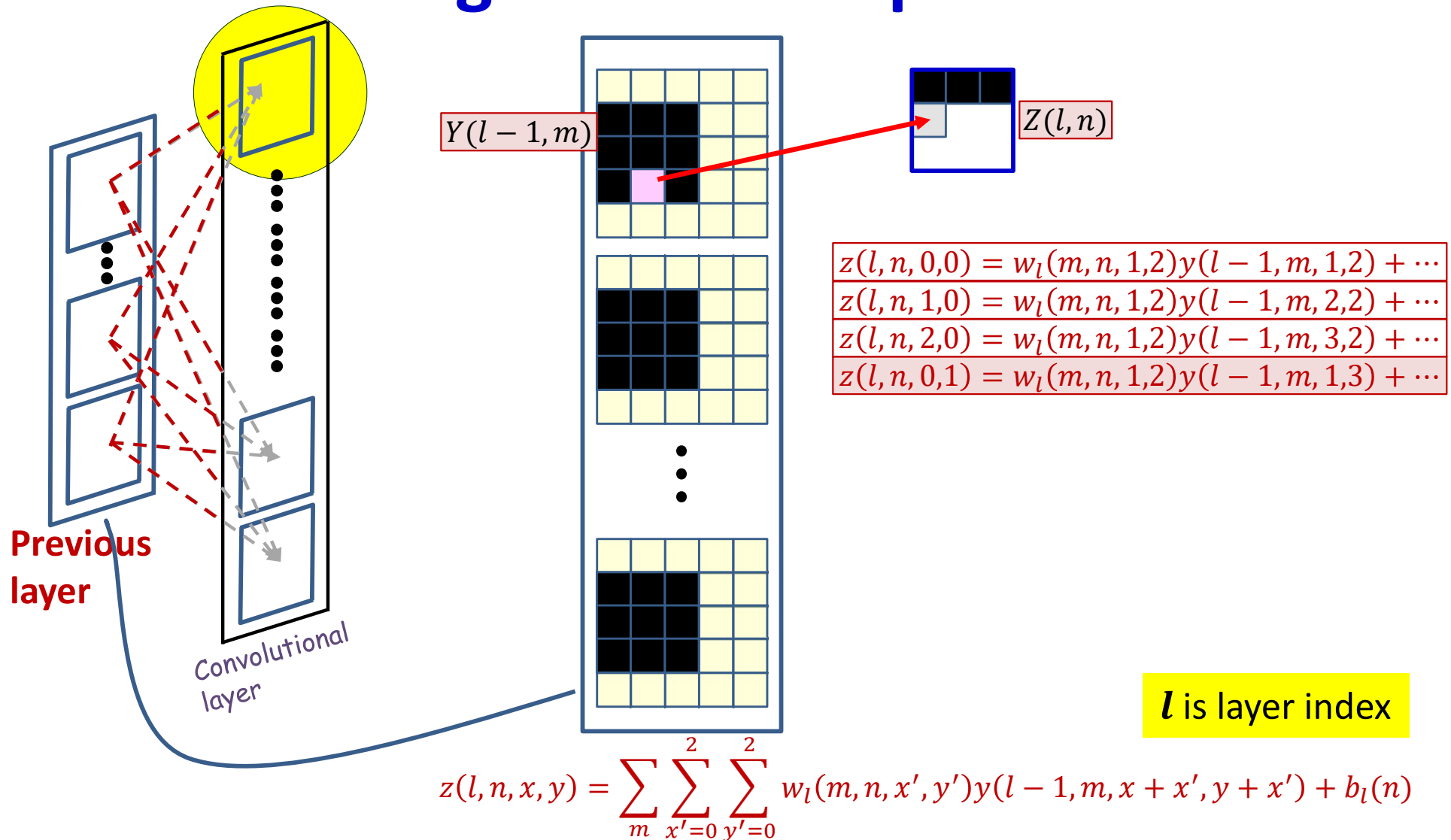
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Convolution: the contribution of a single filter component



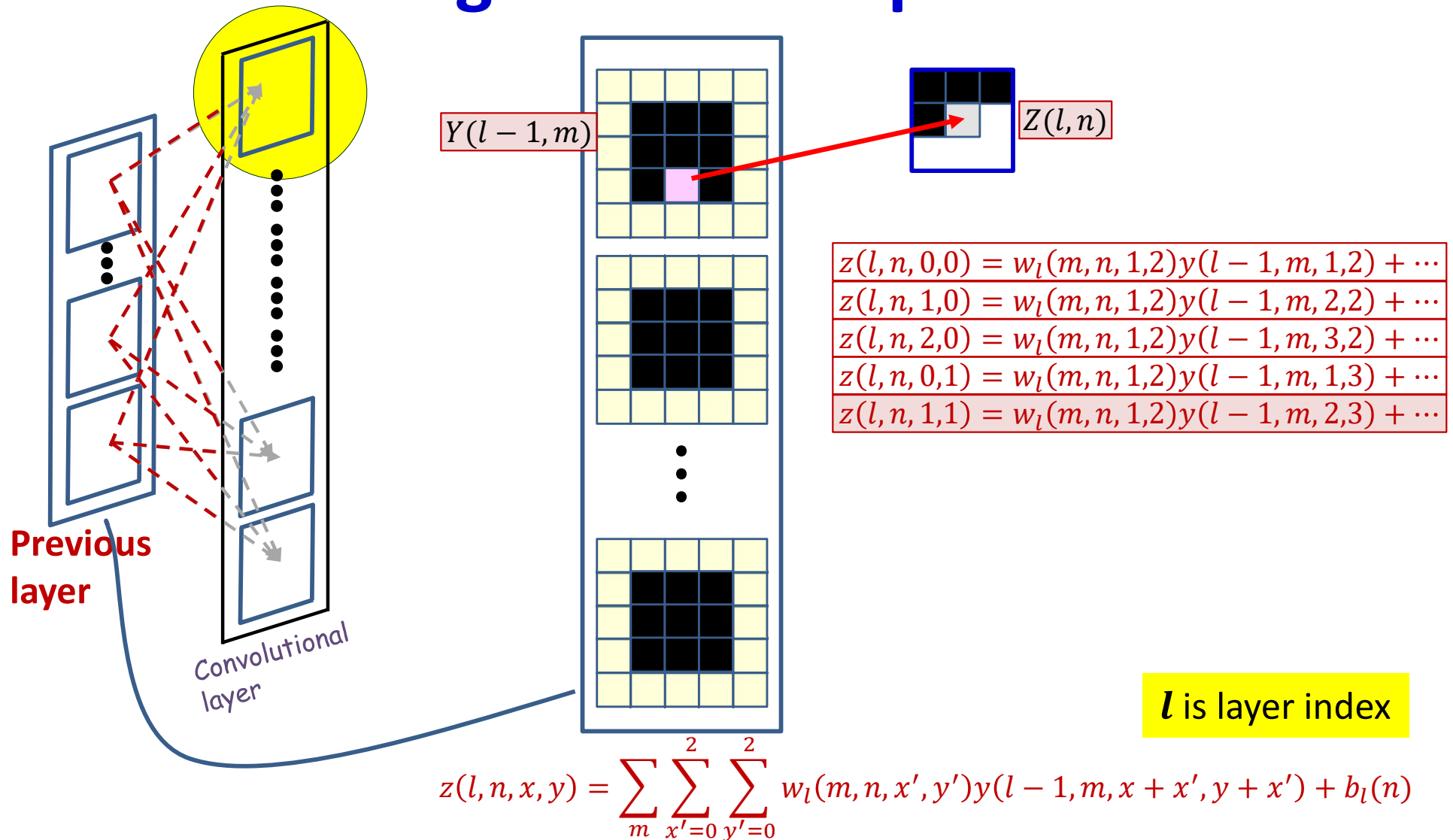
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Convolution: the contribution of a single filter component



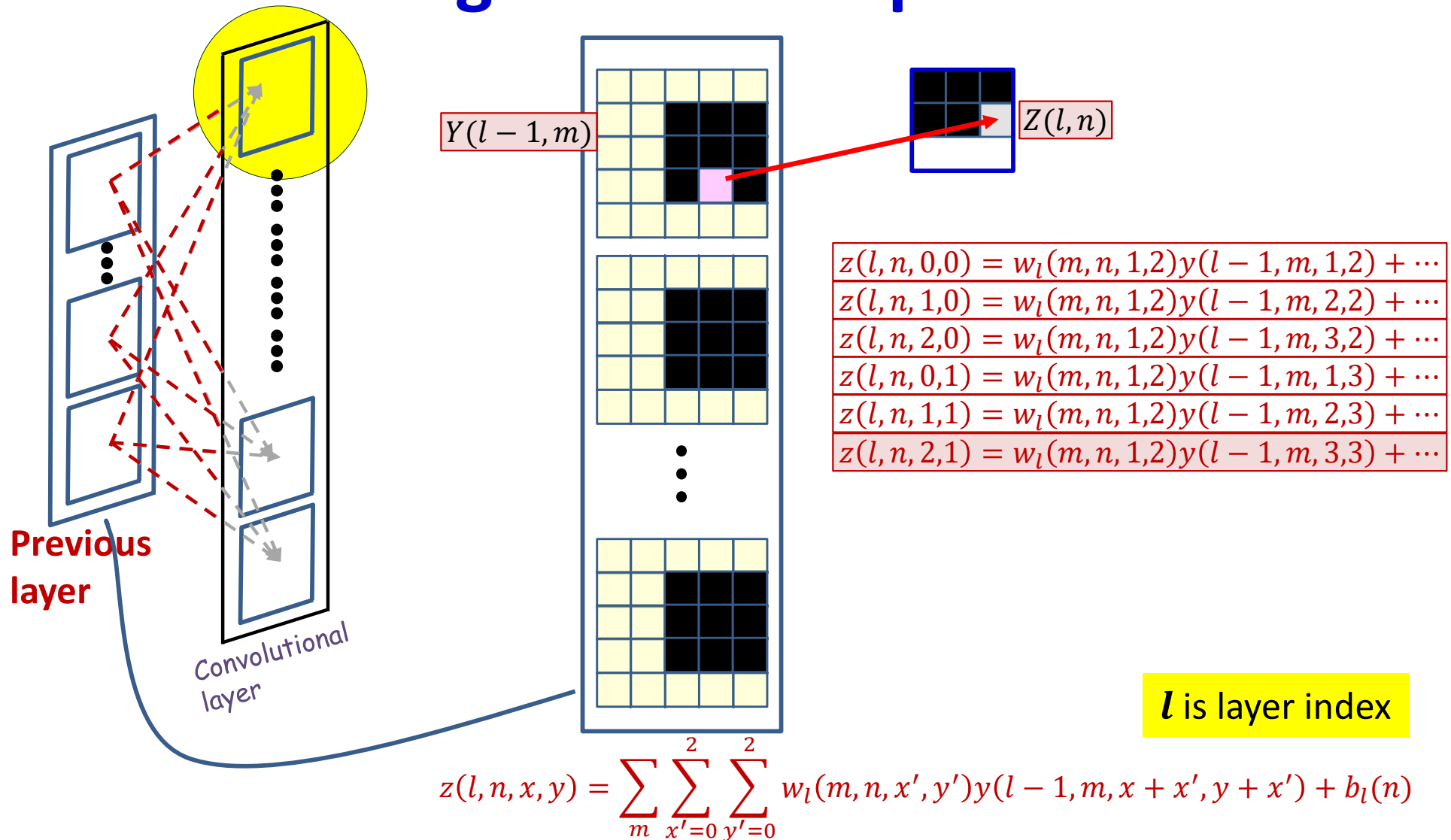
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Convolution: the contribution of a single filter component



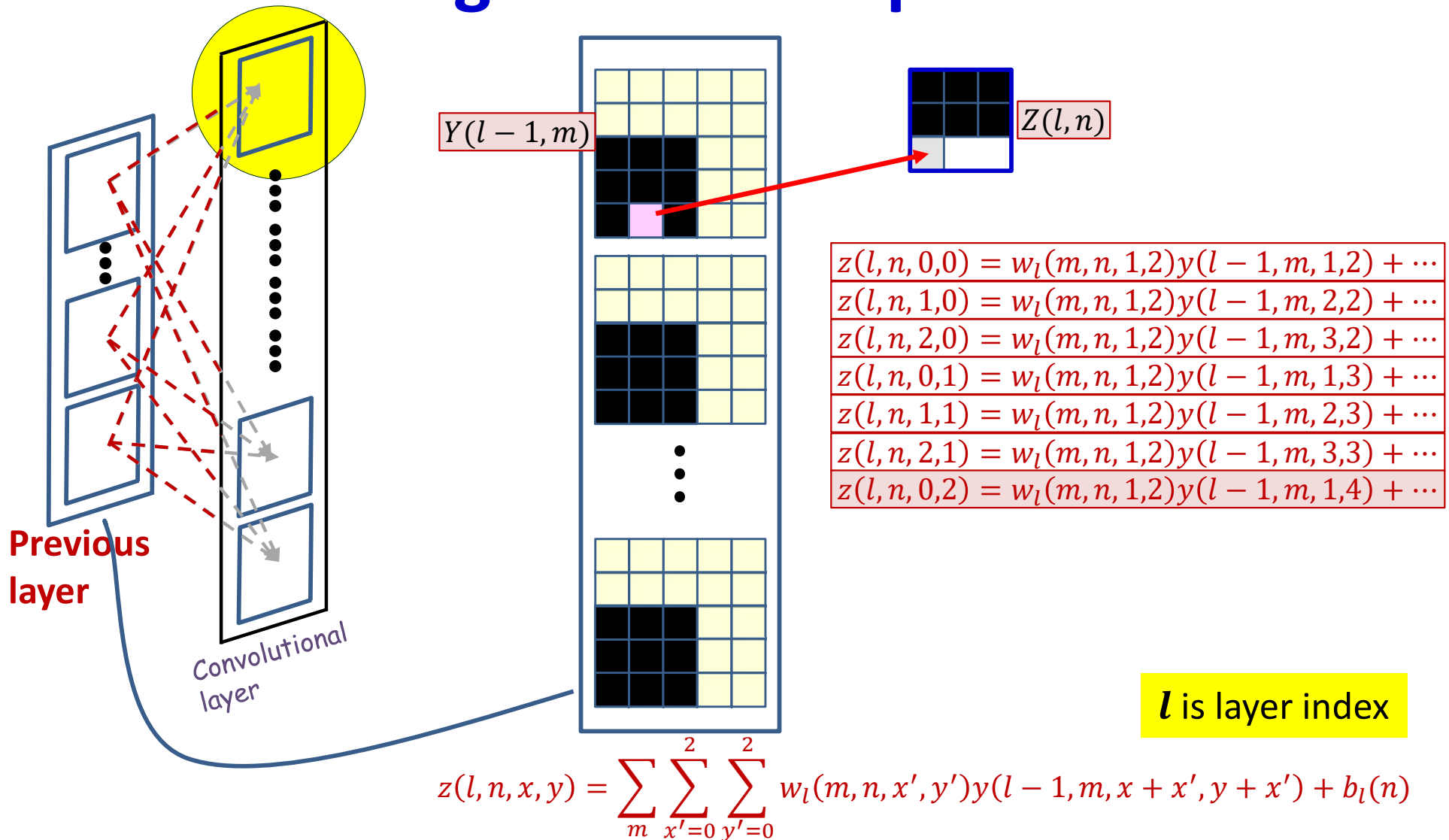
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Convolution: the contribution of a single filter component



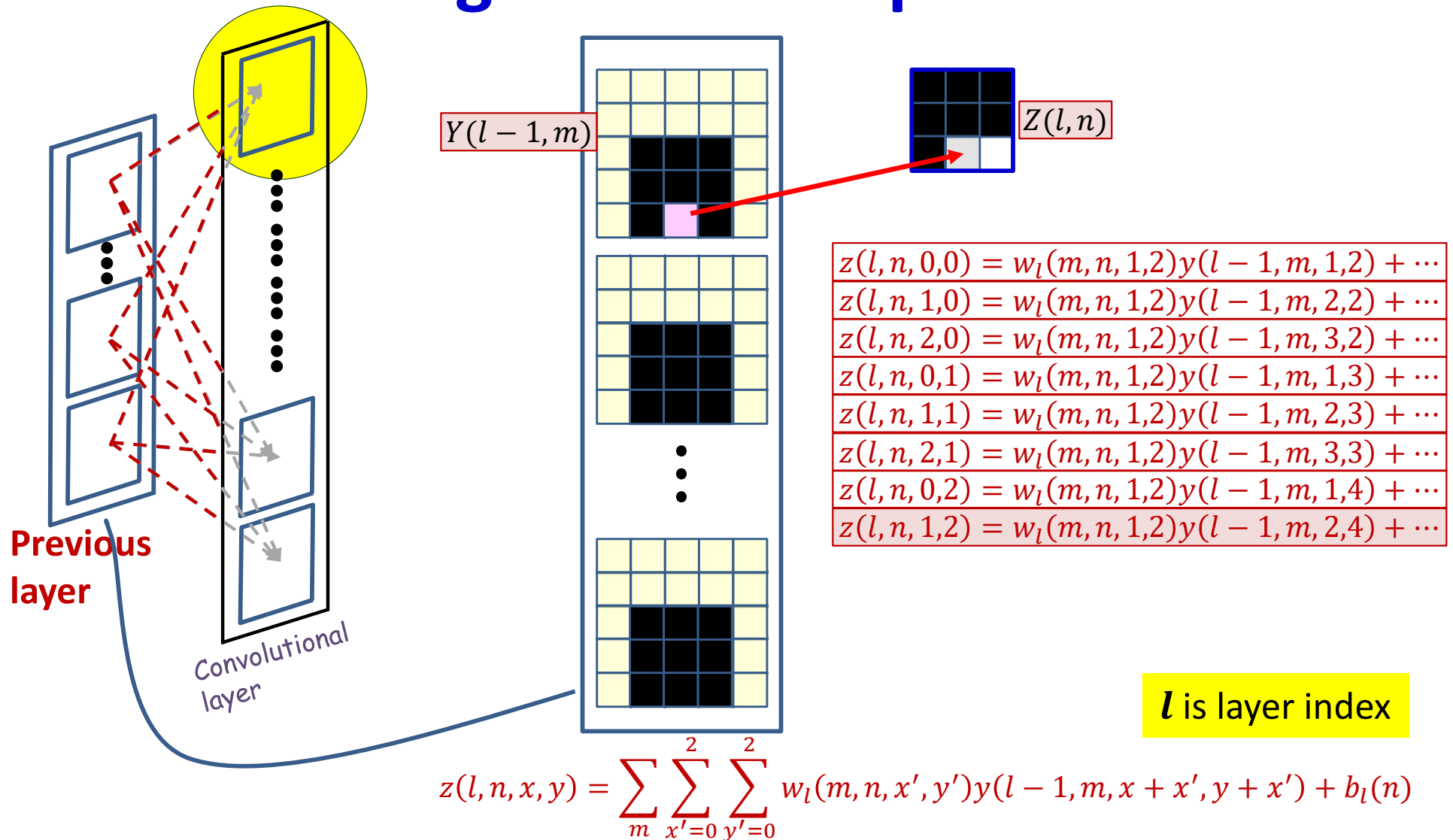
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Convolution: the contribution of a single filter component



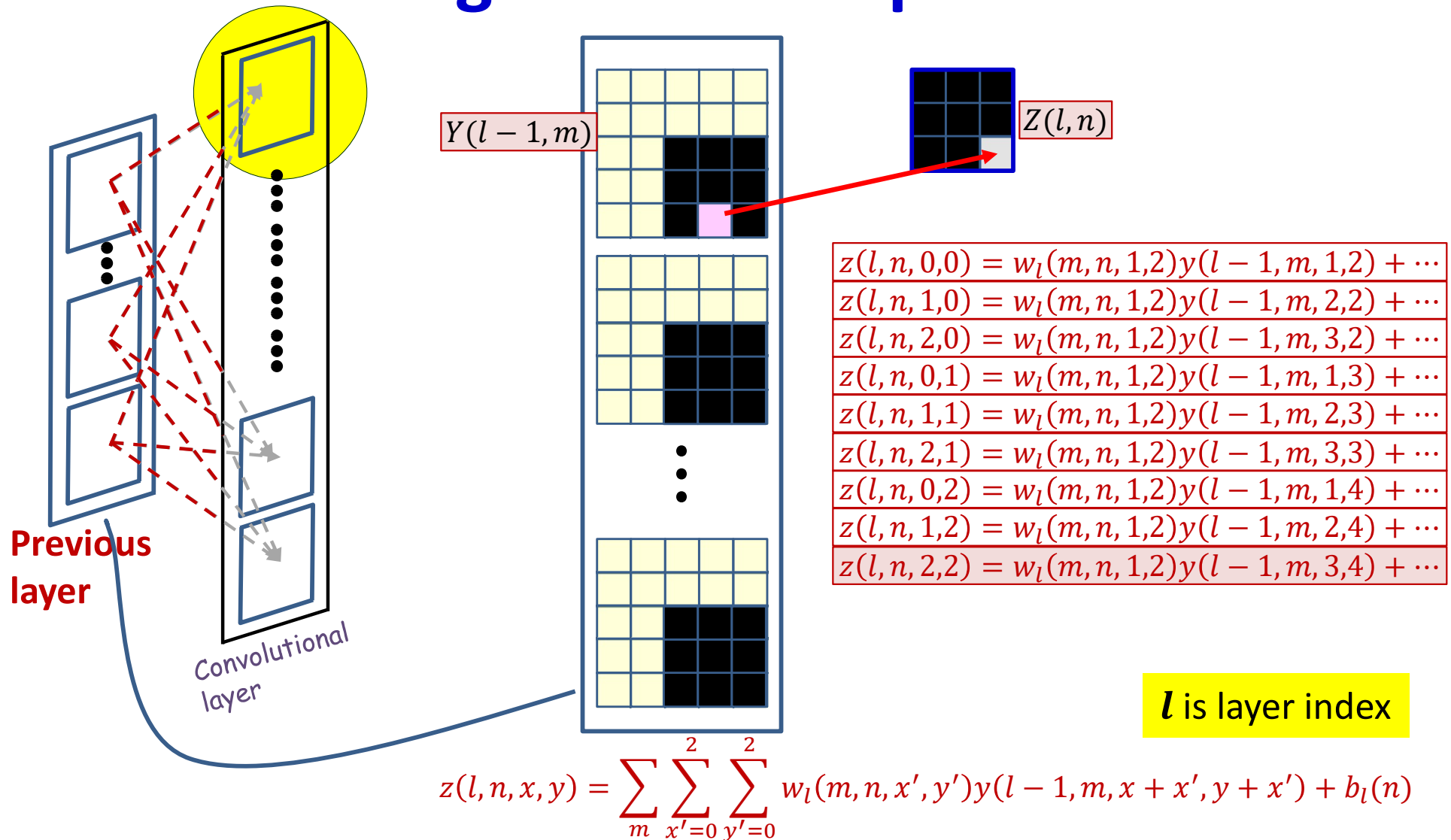
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Convolution: the contribution of a single filter component



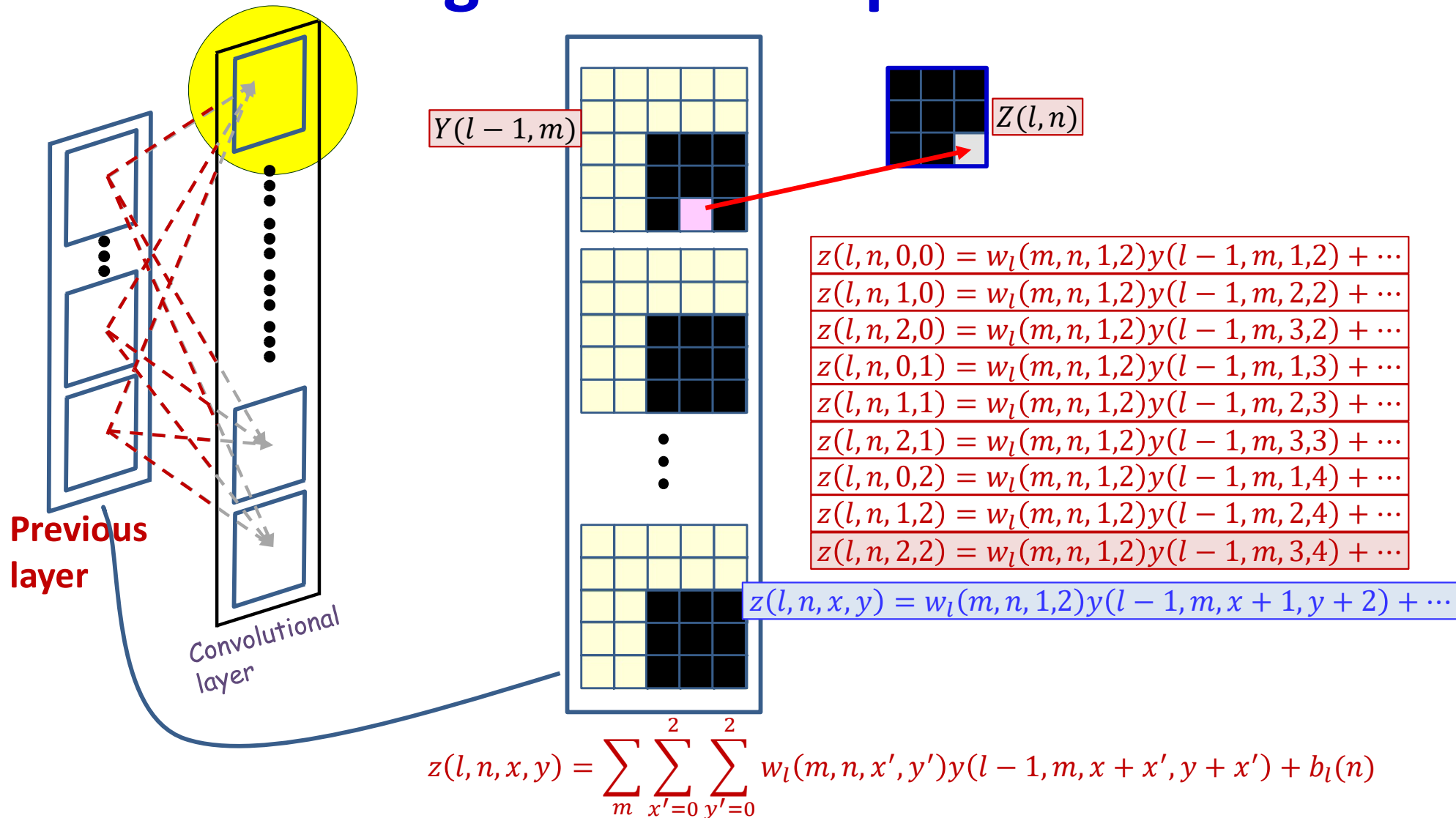
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Convolution: the contribution of a single filter component



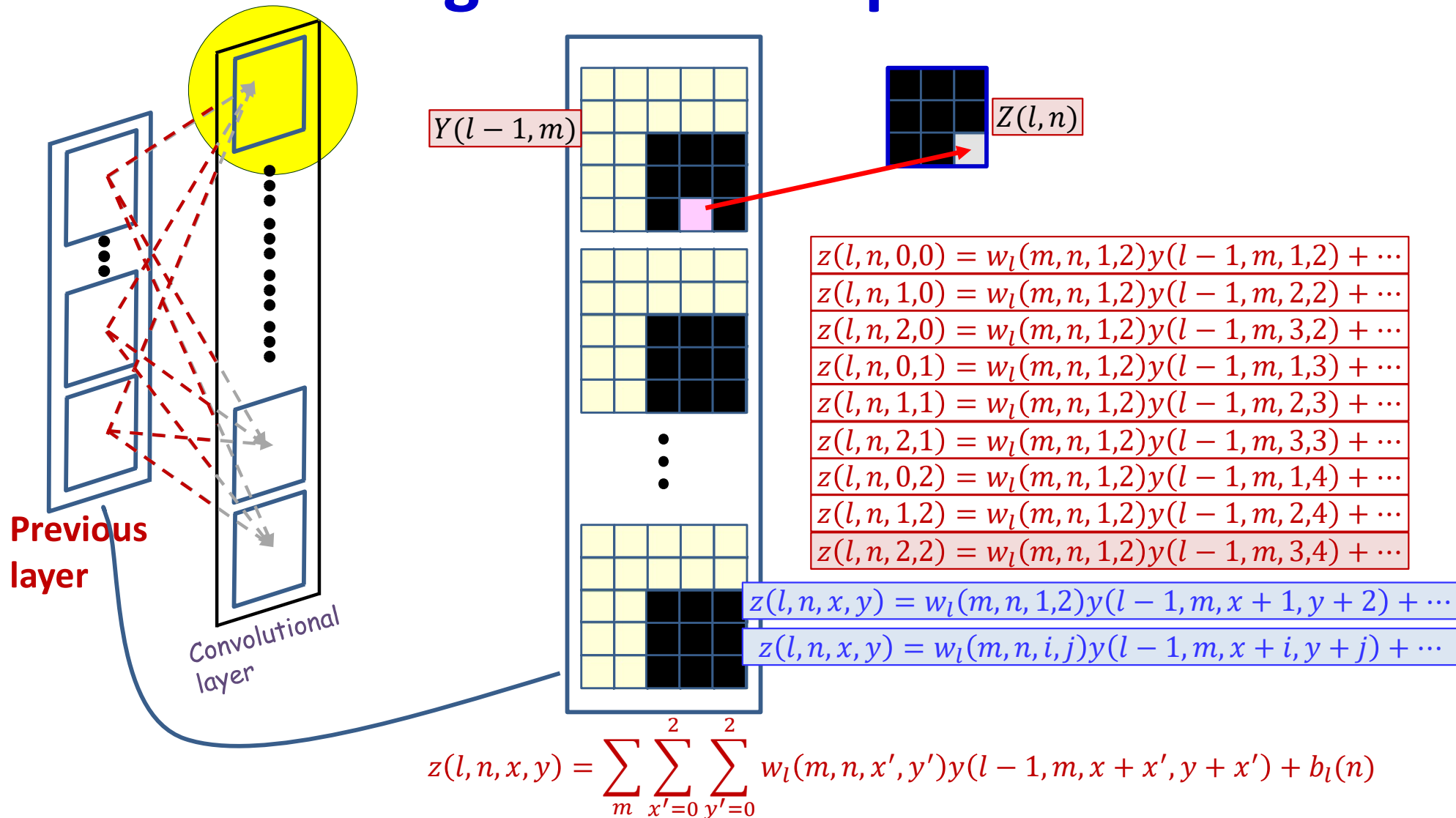
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Convolution: the contribution of a single filter component



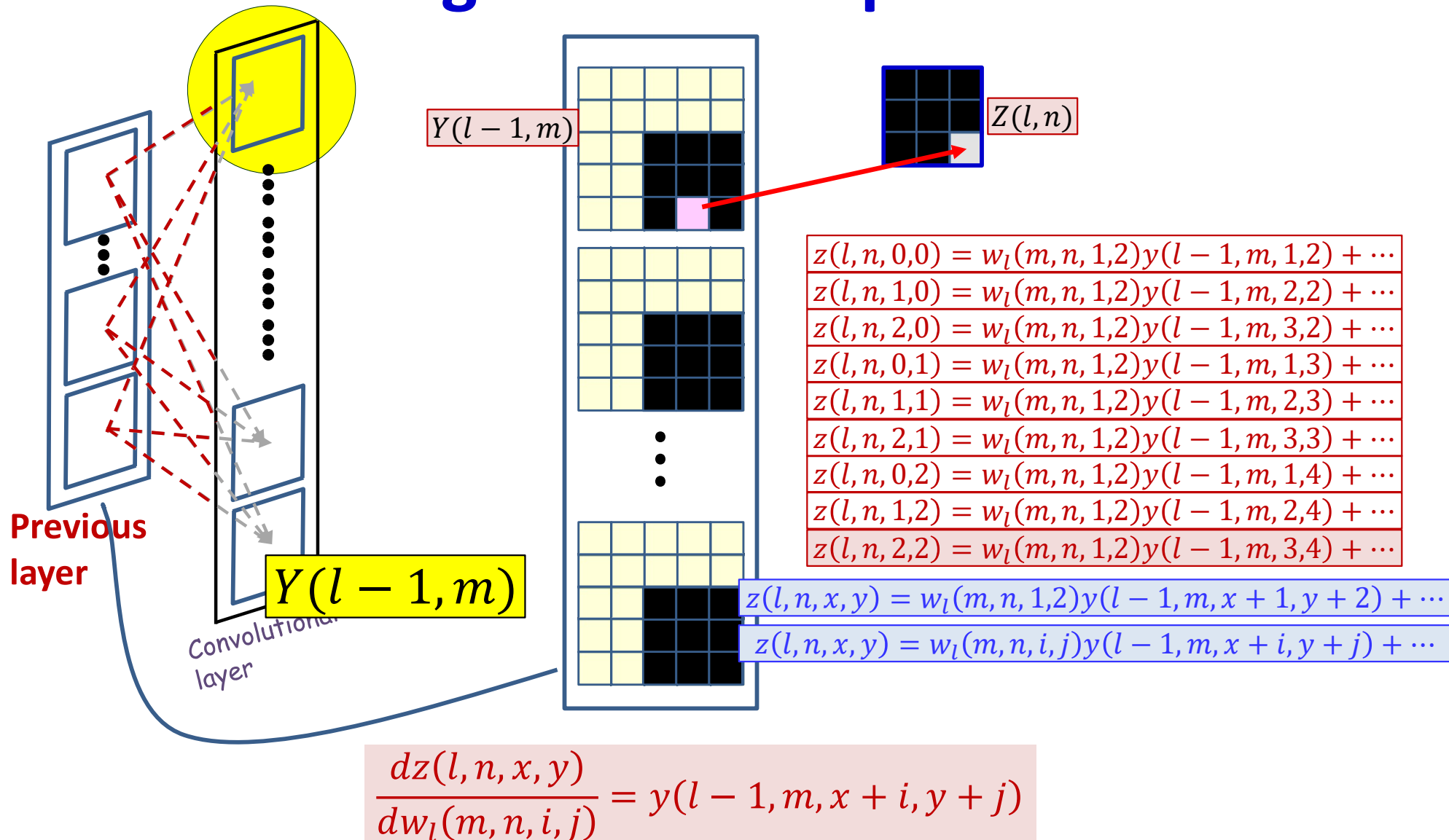
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Convolution: the contribution of a single filter component

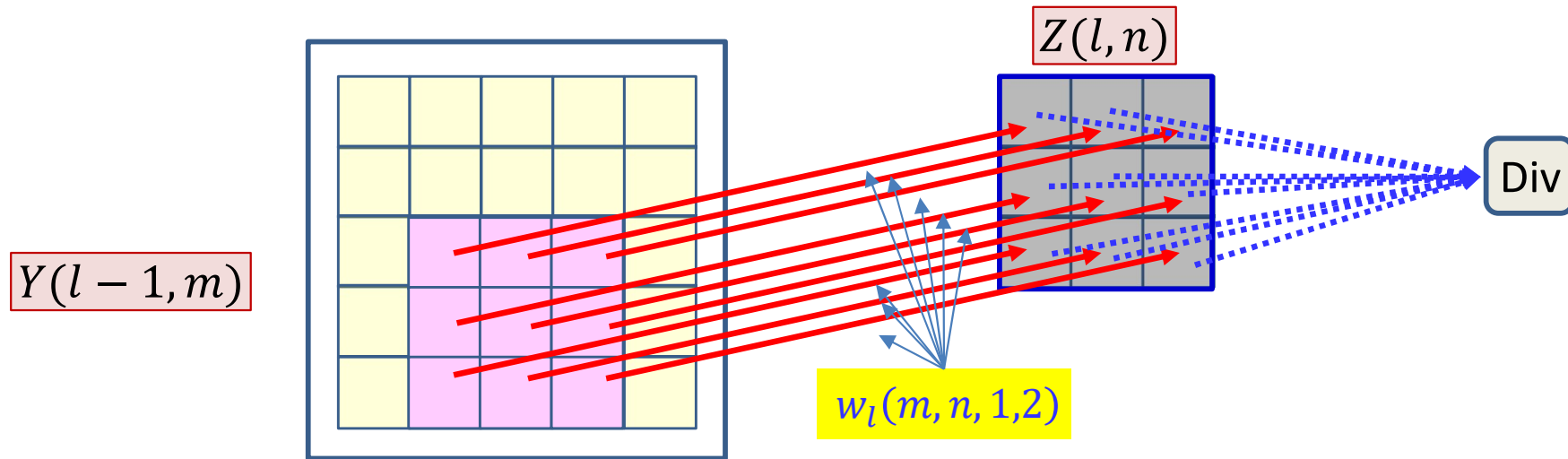


- Each filter component $w_l(m, n, i, j)$ affects several $z(l, n, x, y)$ in the n th output map
 - Consider the contribution of one filter component: e.g. $w_l(m, n, 1, 2)$

Convolution: the contribution of a single filter component



The derivative for a single filter component



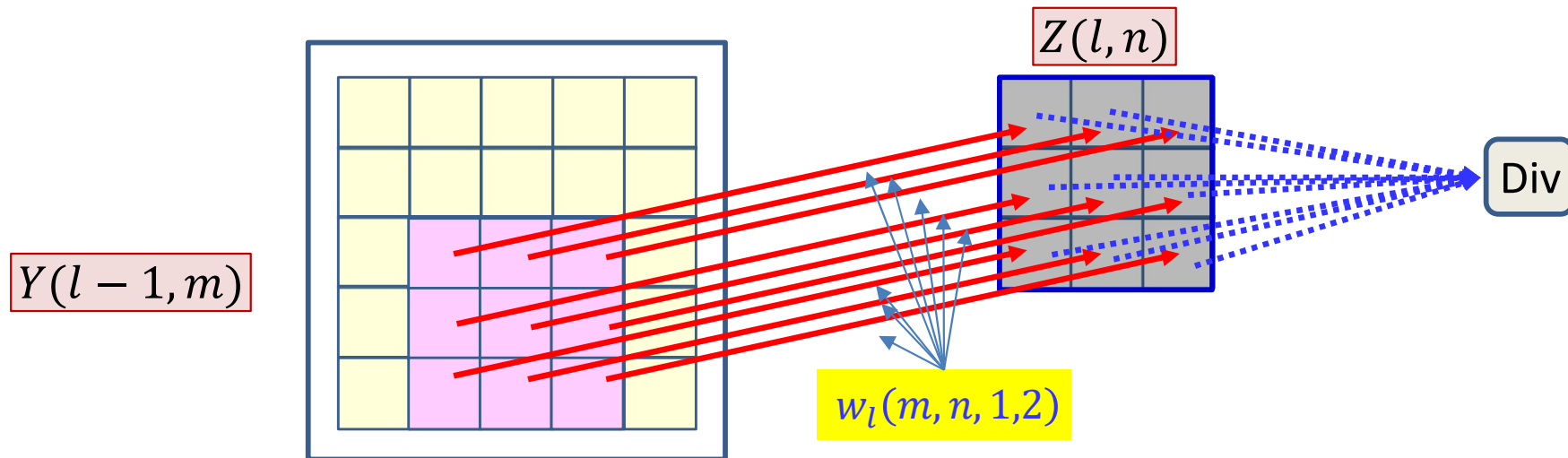
- Each filter component $w_l(m, n, i, j)$ affects several $z(l, n, x, y)$
 - The derivative of each $z(l, n, x, y)$ w.r.t. $w_l(m, n, i, j)$ is given by

$$\frac{dz(l, n, x, y)}{dw_l(m, n, i, j)} = y(l-1, m, x+i, y+j)$$

- The final divergence is influenced by every $z(l, n, x, y)$
- The derivative of the divergence w.r.t $w_l(m, n, i, j)$ must sum over all $z(l, n, x, y)$ terms it influences

$$\frac{dDiv}{dw_l(m, n, i, j)} = \sum_{x,y} \frac{dDiv}{dz(l, n, x, y)} \frac{dz(l, n, x, y)}{dw_l(m, n, i, j)}$$

The derivative for a single filter component



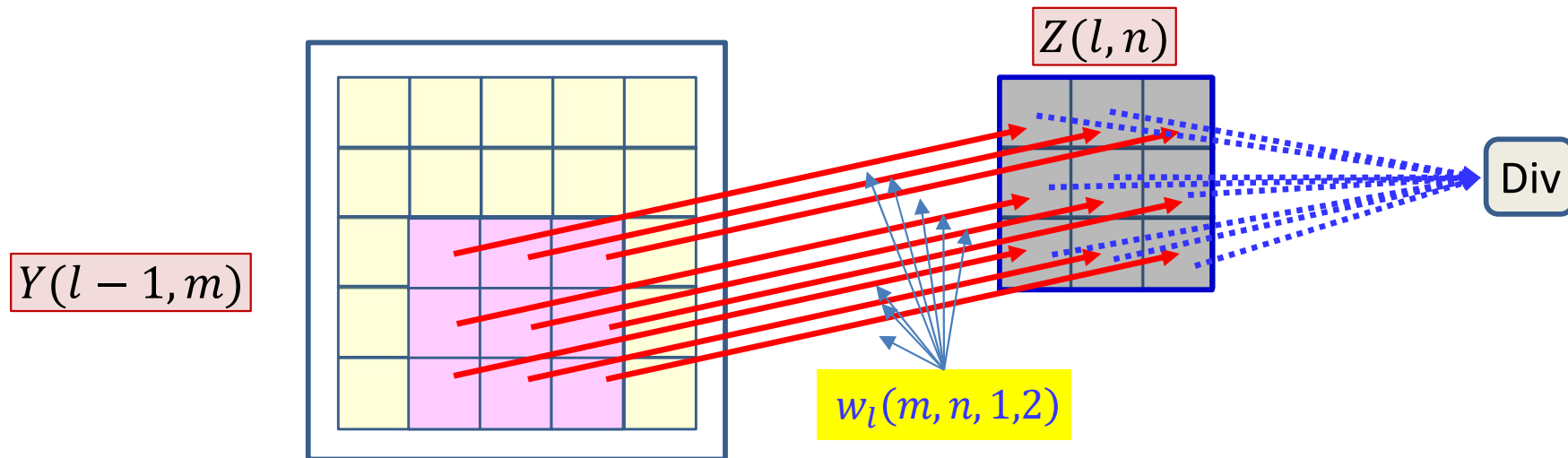
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$$\frac{dDiv}{dw_l(m, n, i, j)} = \sum_{x,y} \left(\frac{dDiv}{dz(l, n, x, y)} \right) \frac{dz(l, n, x, y)}{dw_l(m, n, i, j)}$$

The derivative for a single filter component



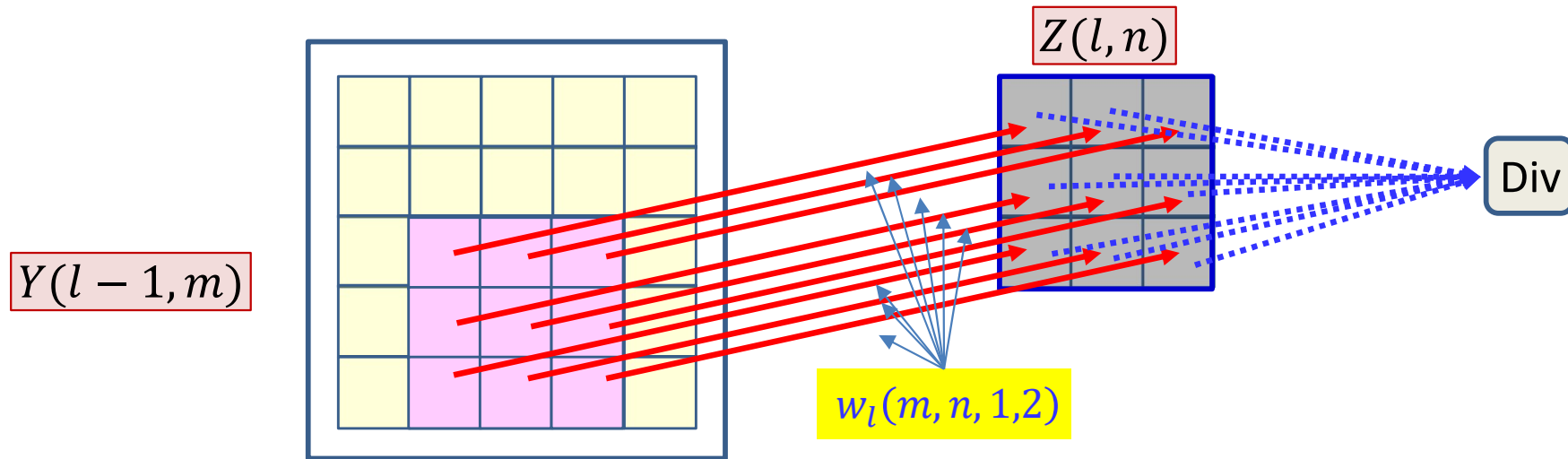
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The derivative for a single filter component



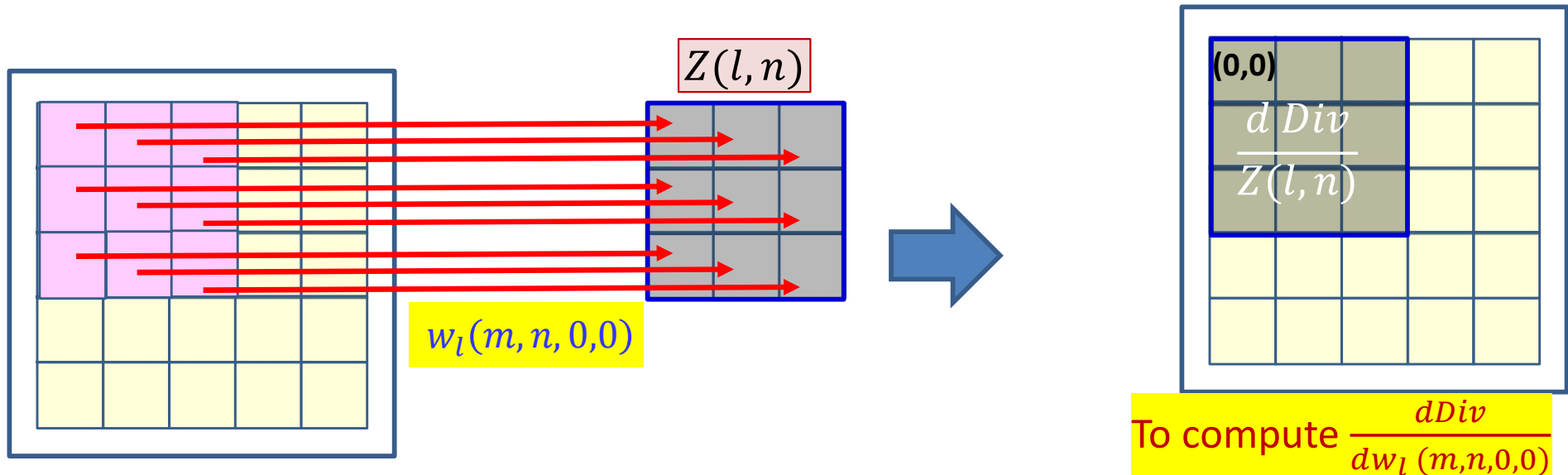
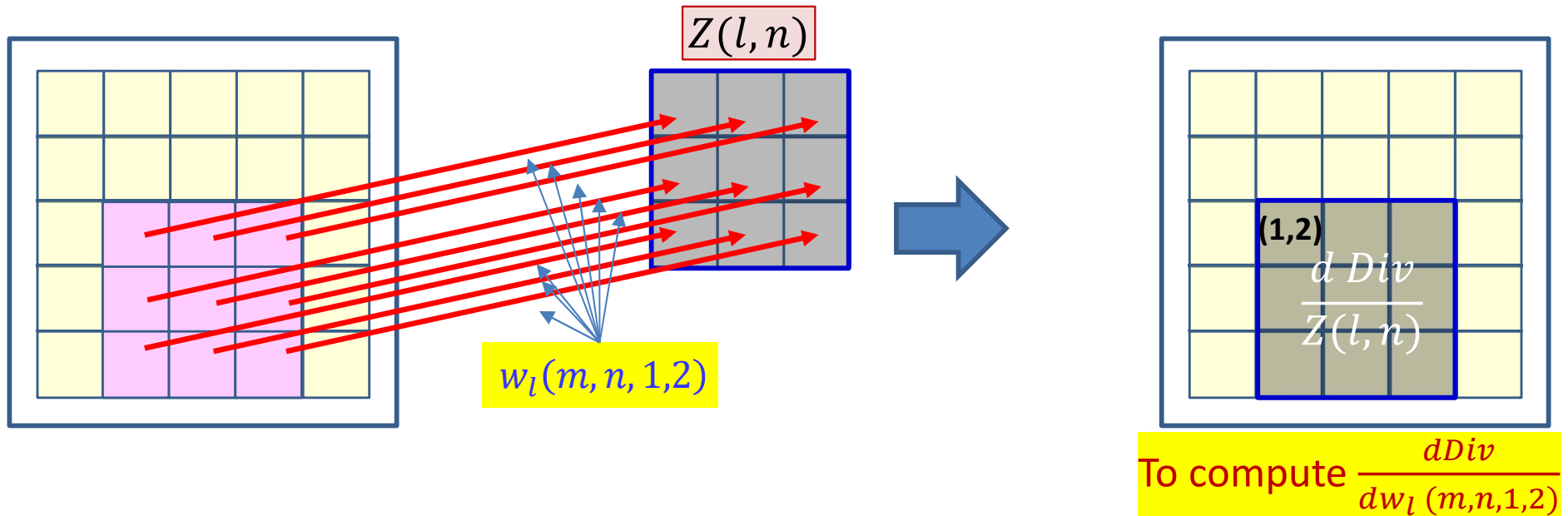
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$$\frac{dDiv}{dw_l(m, n, i, j)} = \sum_{x,y} \frac{dDiv}{dz(l, n, x, y)} y(l-1, m, x+i, y+j)$$

The derivative for a single filter component



But this too is a convolution

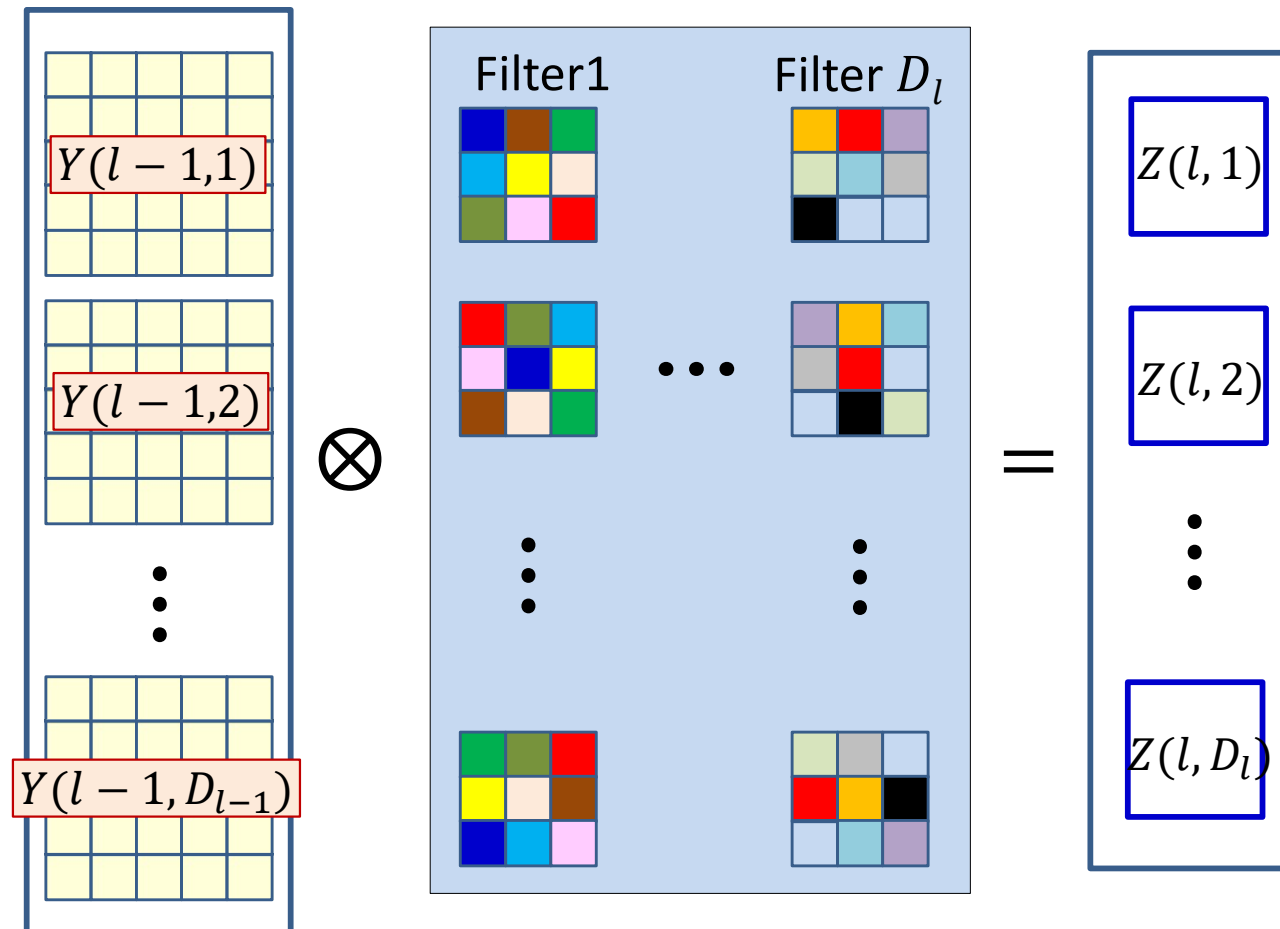
$$\frac{dDiv}{dw_l(m, n, i, j)} = \sum_{x, y} \frac{dDiv}{dz(l, n, x, y)} y(l-1, m, x+i, y+j)$$

- The derivatives for all components of all filters can be computed directly from the above formula
 - To compute the derivative for $w_l(m, n, i, j)$, “place” the $dDiv/dz(l, n)$ map on $y(l-1, m)$ map positioned at (i, j) and compute the inner product
- In fact, it is just a convolution

$$\frac{dDiv}{dw_l(m, n, i, j)} = \frac{dDiv}{dz(l, n)} \otimes y(l-1, m)$$

- How?

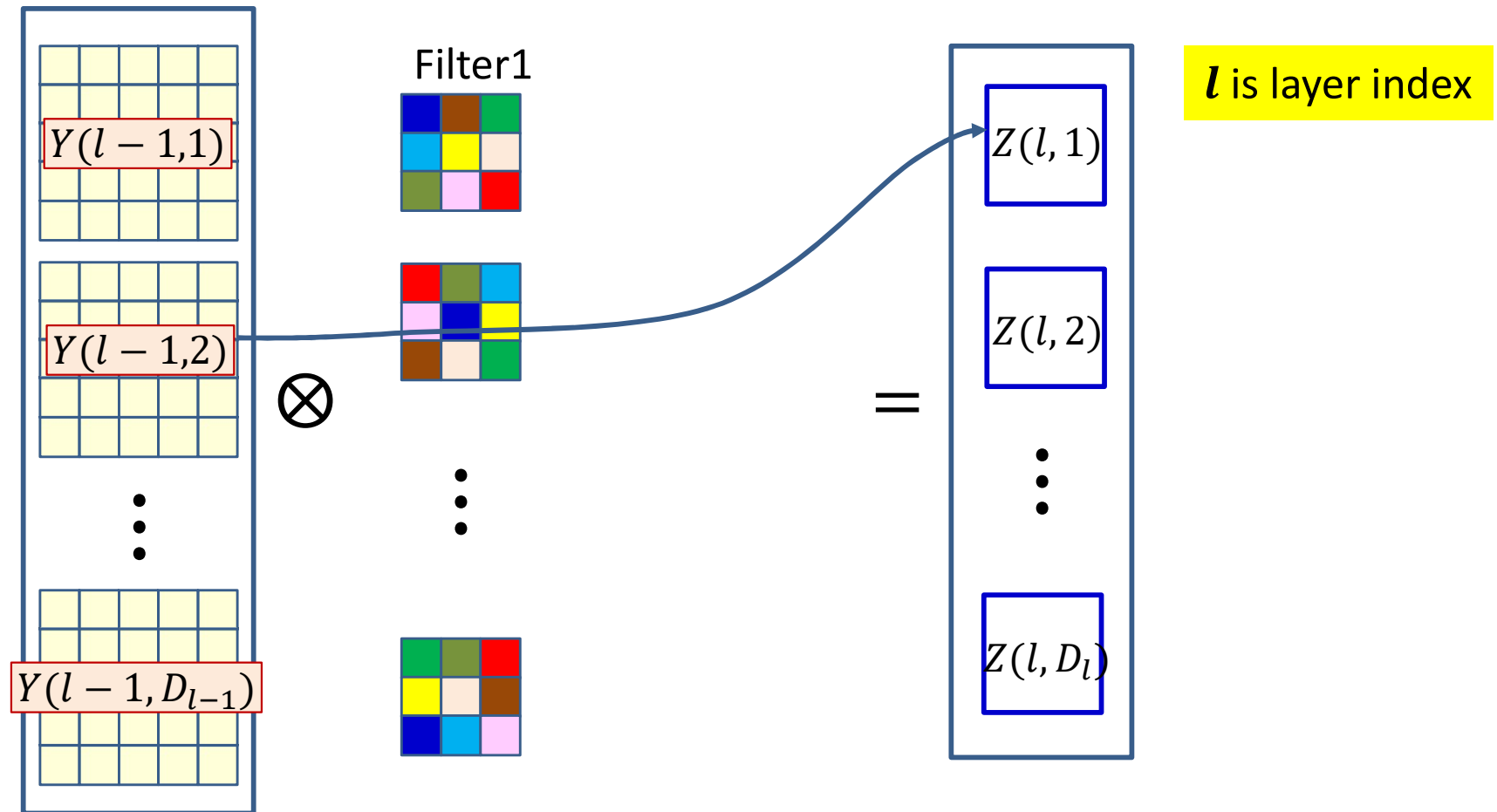
Recap: Convolution



$$z(l, n, x, y) = \sum_m \sum_{i=0}^2 \sum_{j=0}^2 w_l(m, n, i, j) y(l-1, m, x+i, y+j) + b_l(n)$$

- Forward computation: Each filter produces an affine map

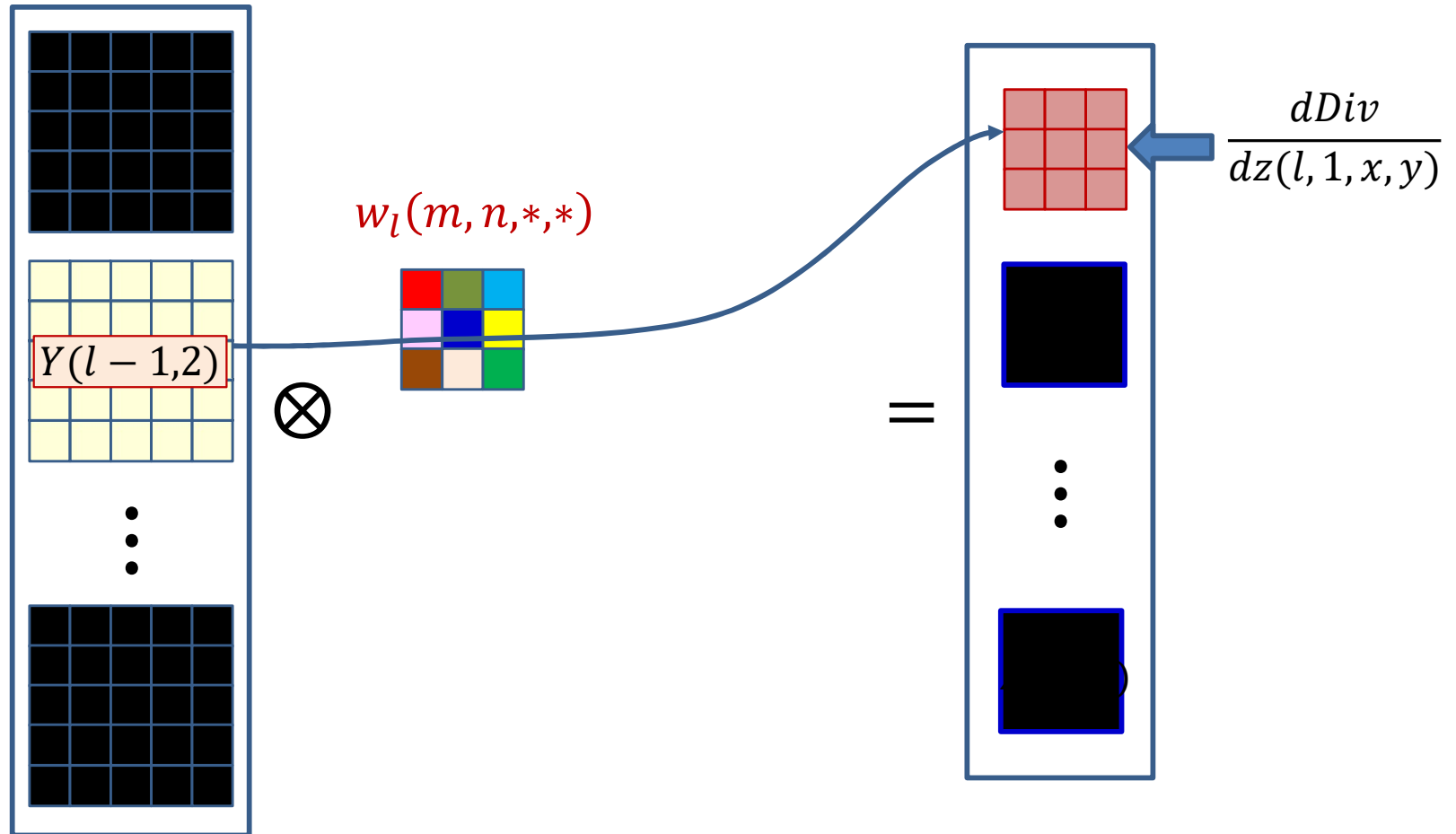
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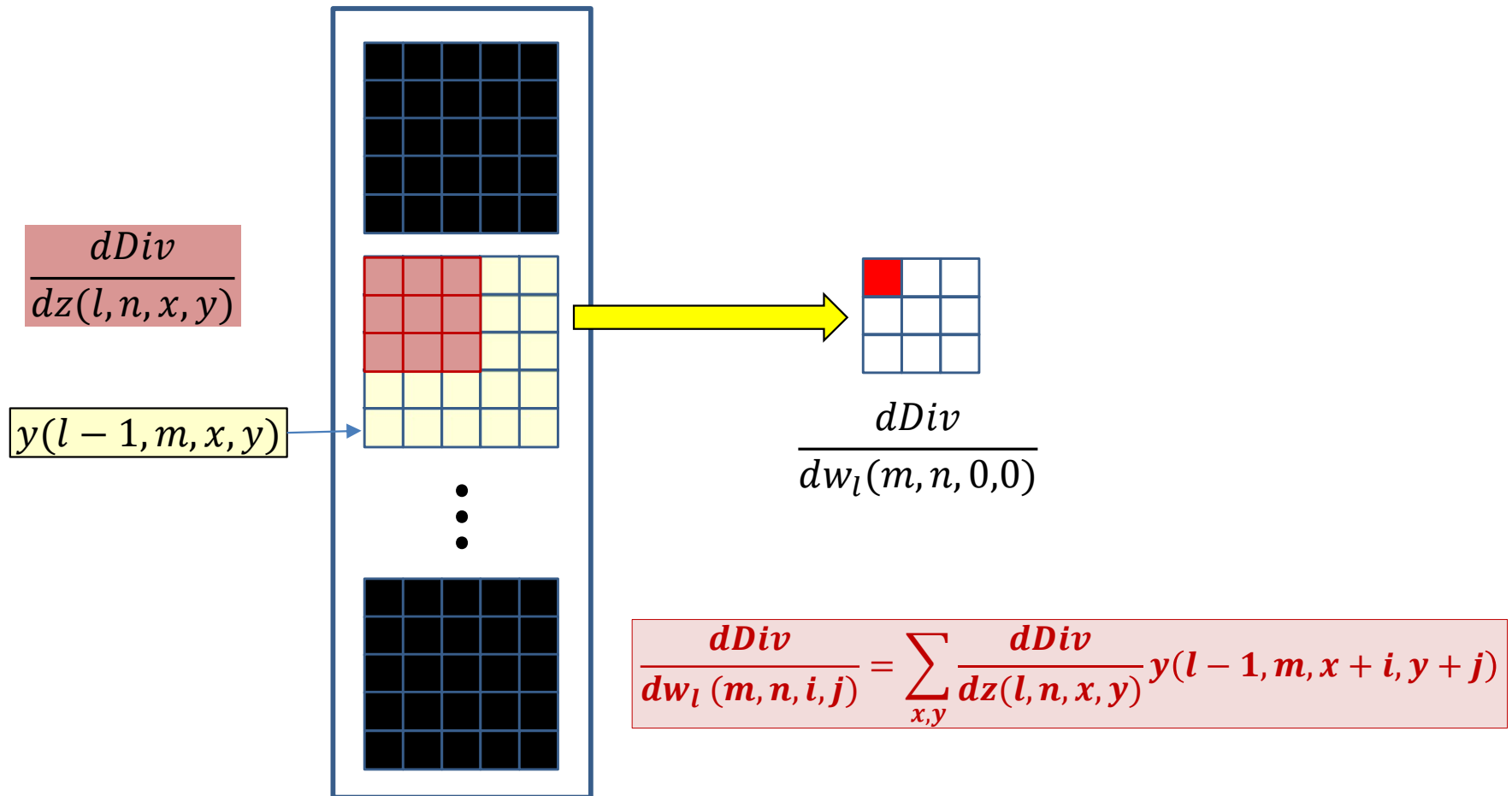
- $Y(l-1, m)$ influences $Z(l, n)$ through $w_l(m, n)$

The filter derivative



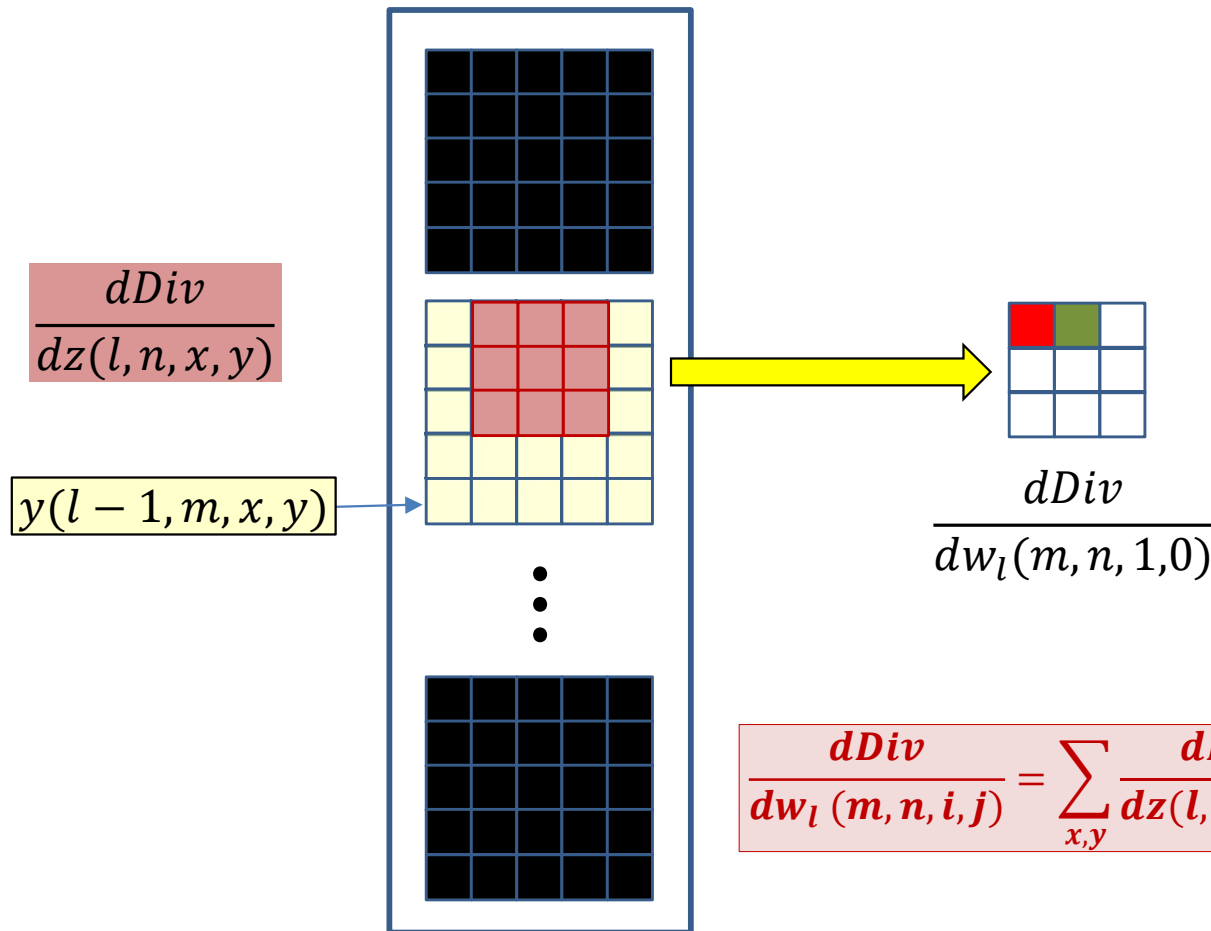
- The derivatives of the divergence w.r.t. every element of $Z(l, n)$ is known
 - Must use them to compute the derivative for $w_l(m, n, *, *)$

The filter derivative



- The derivatives of the divergence w.r.t. every element of $Z(l, n)$ is known
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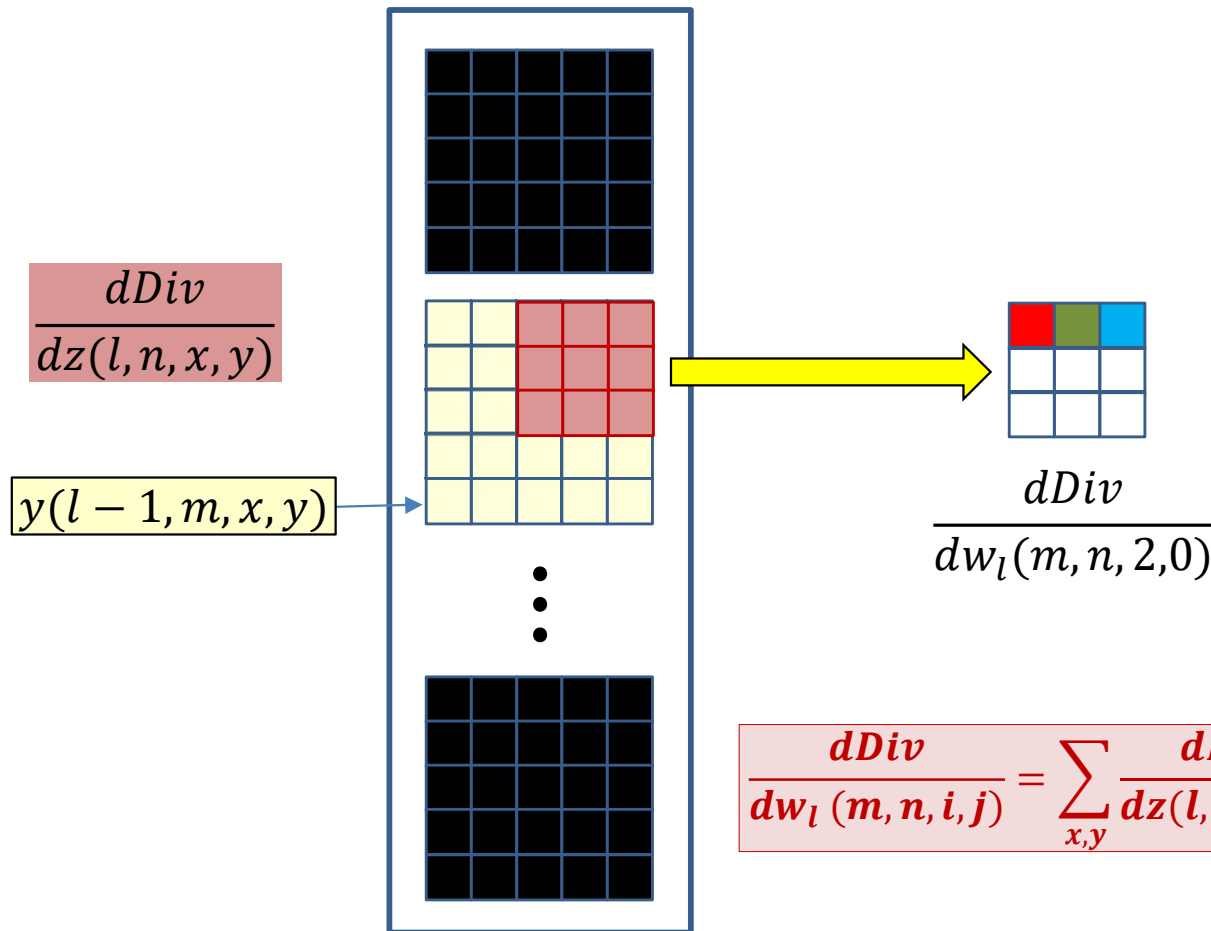
The filter derivative



$$\frac{dDiv}{dw_l(m, n, i, j)} = \sum_{x, y} \frac{dDiv}{dz(l, n, x, y)} y(l-1, m, x+i, y+j)$$

- The derivatives of the divergence w.r.t. every element of $Z(l, n)$ is known
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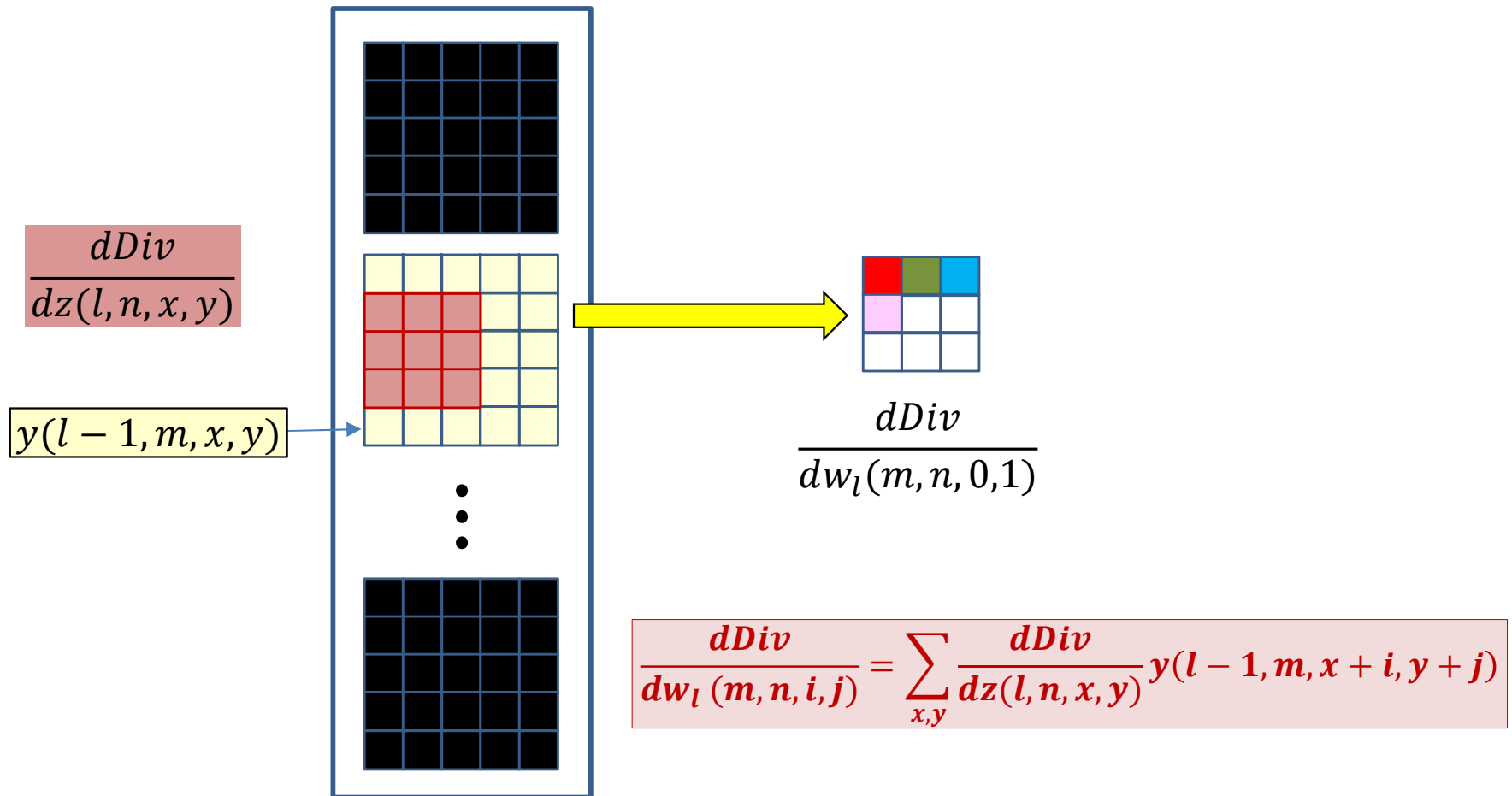
The filter derivative



$$\frac{dDiv}{dw_l(m, n, i, j)} = \sum_{x,y} \frac{dDiv}{dz(l, n, x, y)} y(l-1, m, x+i, y+j)$$

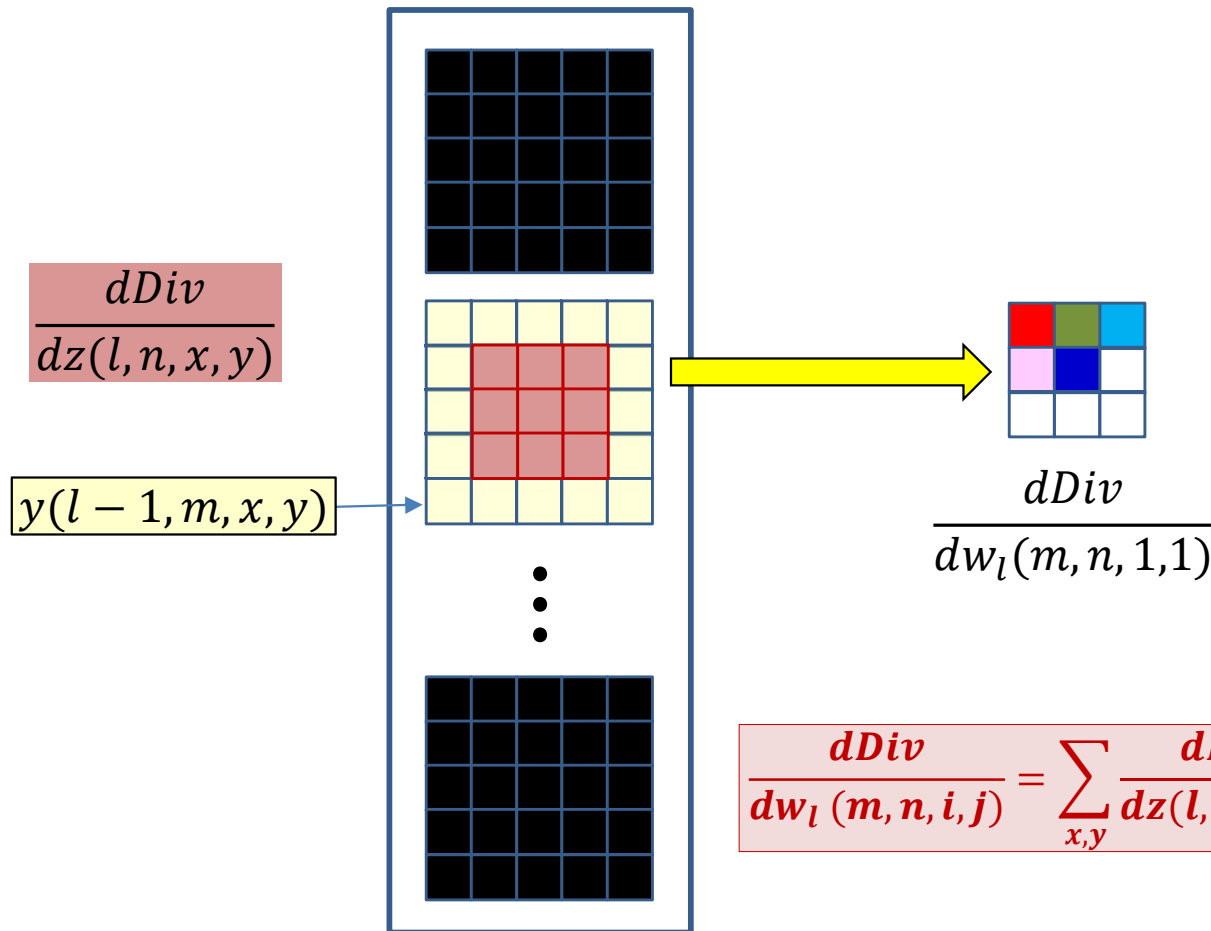
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The filter derivative



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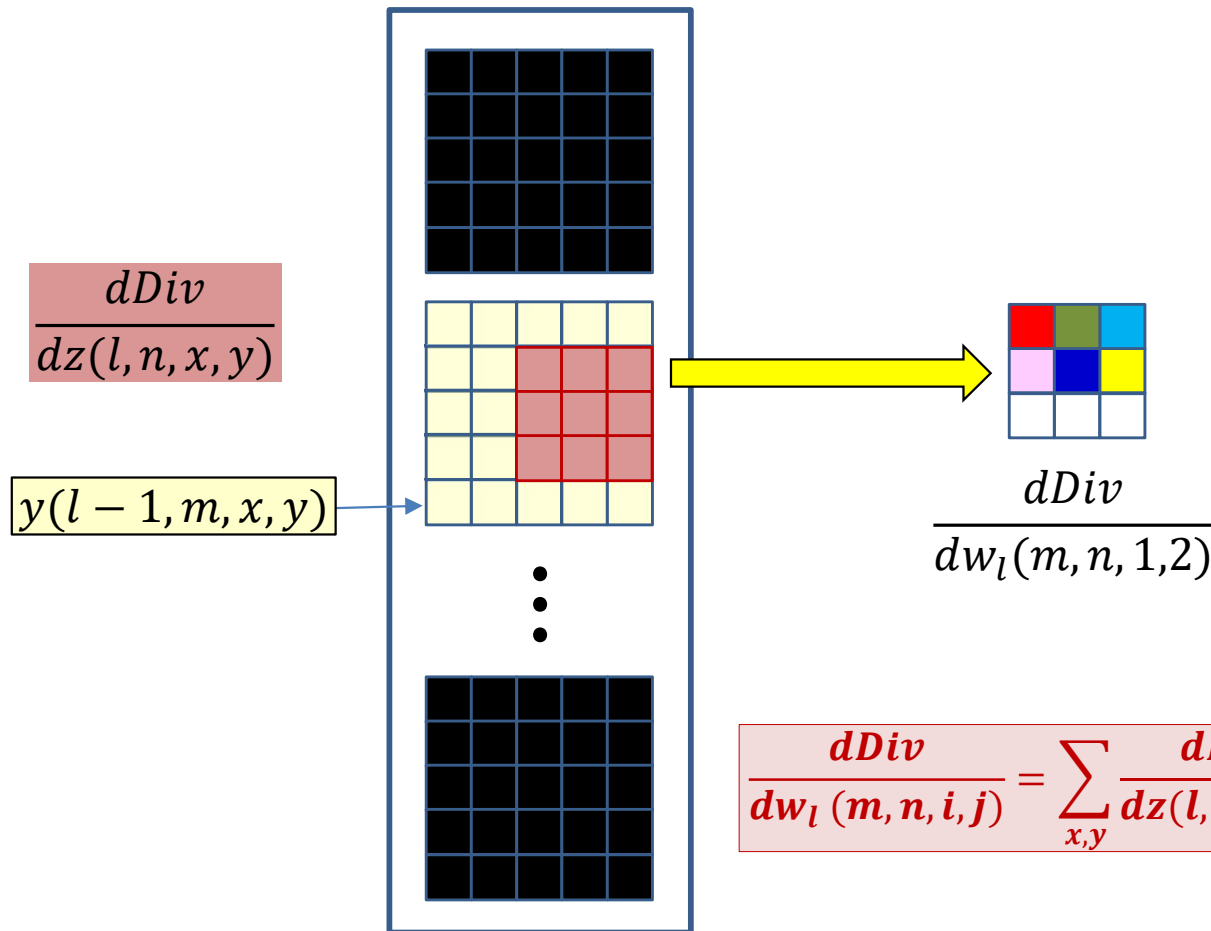
The filter derivative



$$\frac{dDiv}{dw_l(m, n, i, j)} = \sum_{x,y} \frac{dDiv}{dz(l, n, x, y)} y(l-1, m, x+i, y+j)$$

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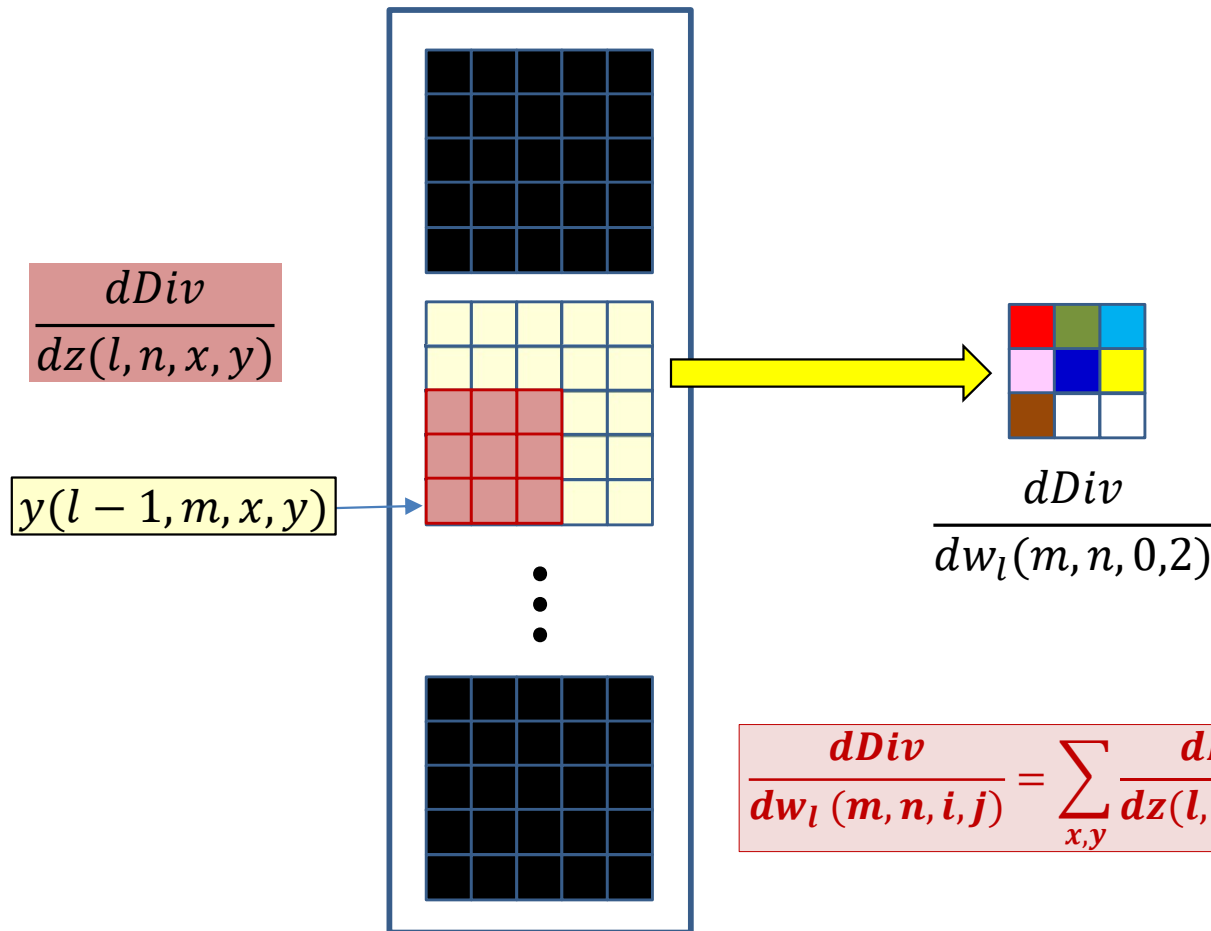
The filter derivative



$$\frac{dDiv}{dw_l(m, n, i, j)} = \sum_{x, y} \frac{dDiv}{dz(l, n, x, y)} y(l-1, m, x+i, y+j)$$

- The derivatives of the divergence w.r.t. every element of $Z(l, n)$ is known
 - Must use them to compute the derivative for $w_l(m, n, *, *)$

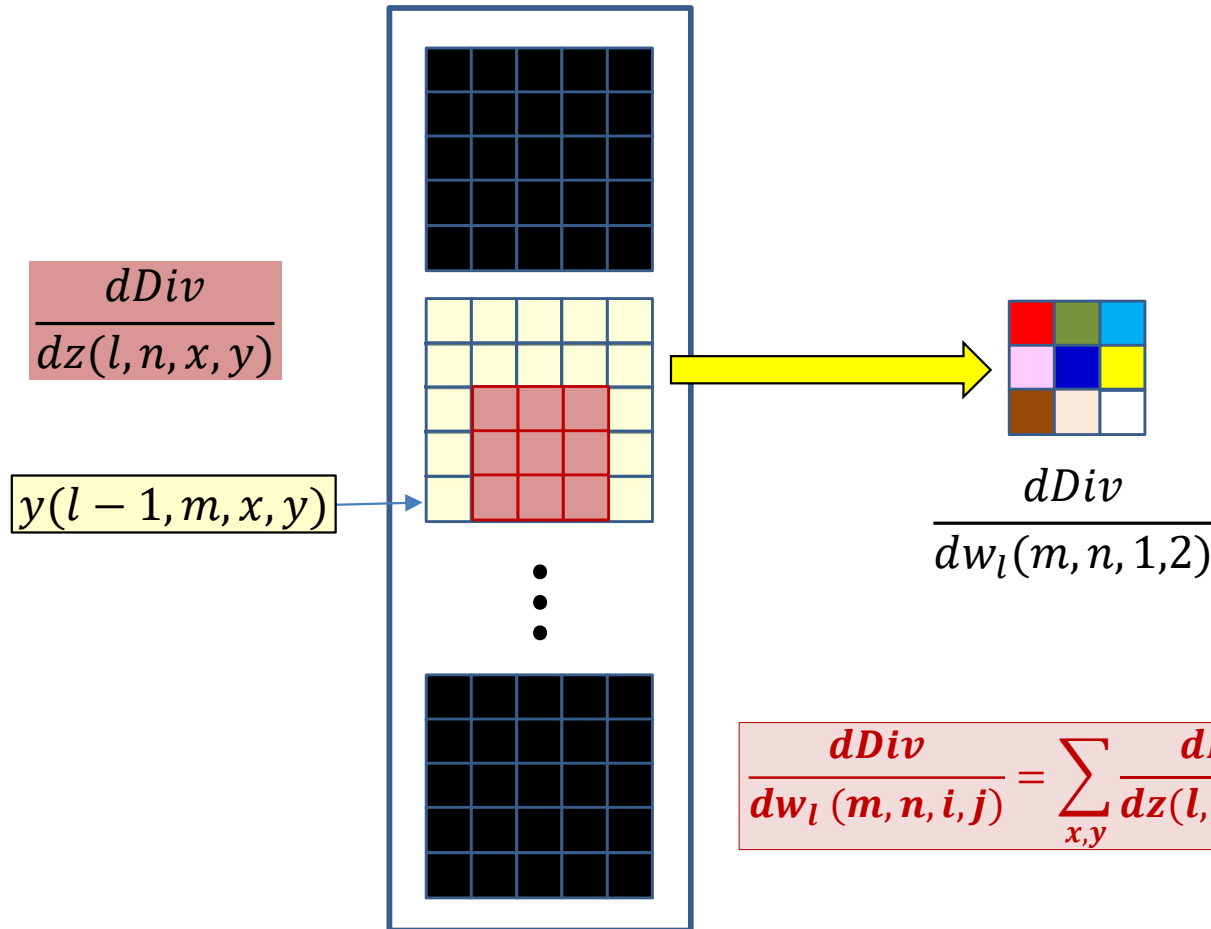
The filter derivative



$$\frac{dDiv}{dw_l(m, n, i, j)} = \sum_{x,y} \frac{dDiv}{dz(l, n, x, y)} y(l-1, m, x+i, y+j)$$

- The derivatives of the divergence w.r.t. every element of $Z(l, n)$ is known
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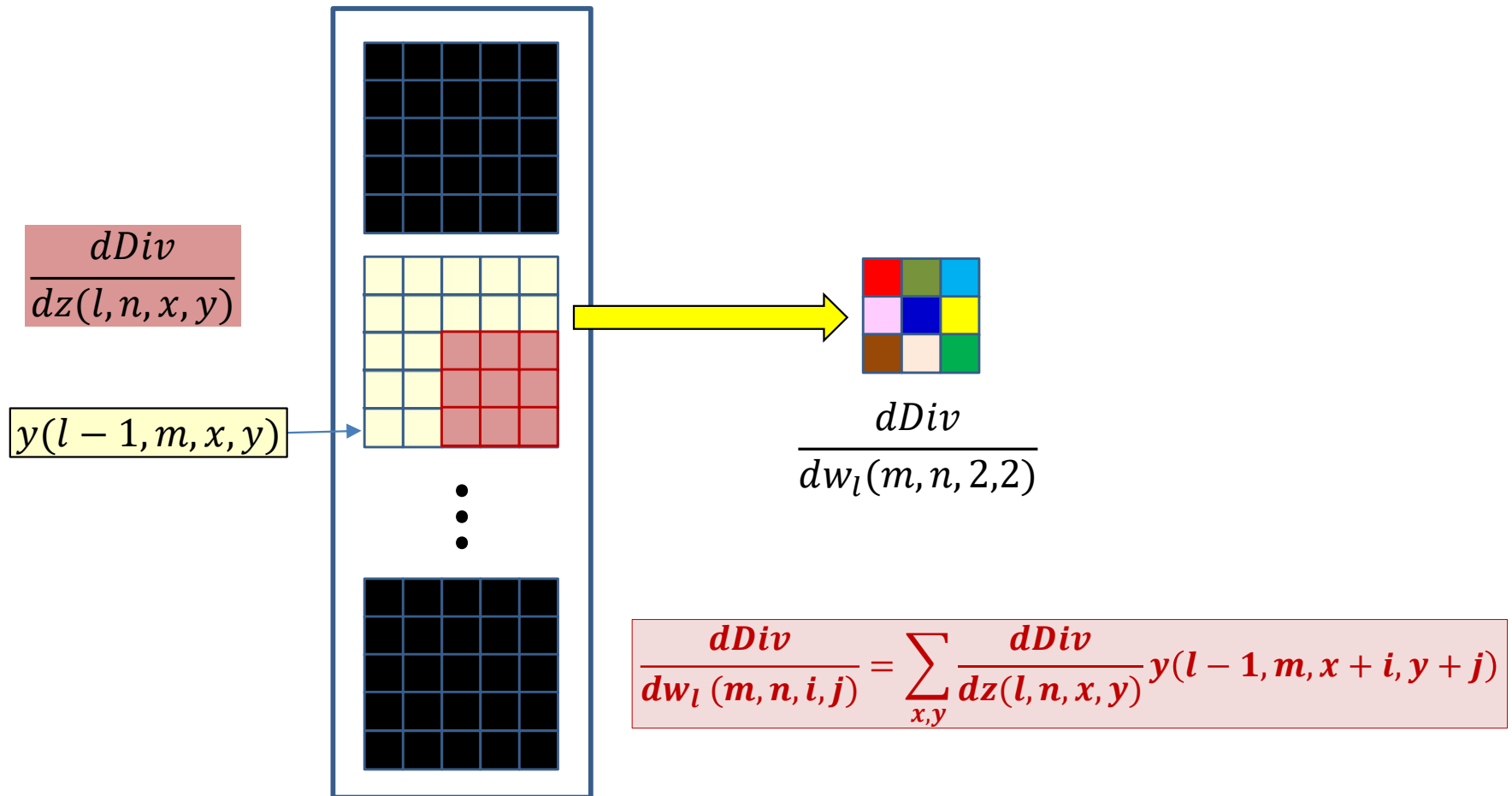
The filter derivative



$$\frac{dDiv}{dw_l(m, n, i, j)} = \sum_{x, y} \frac{dDiv}{dz(l, n, x, y)} y(l-1, m, x+i, y+j)$$

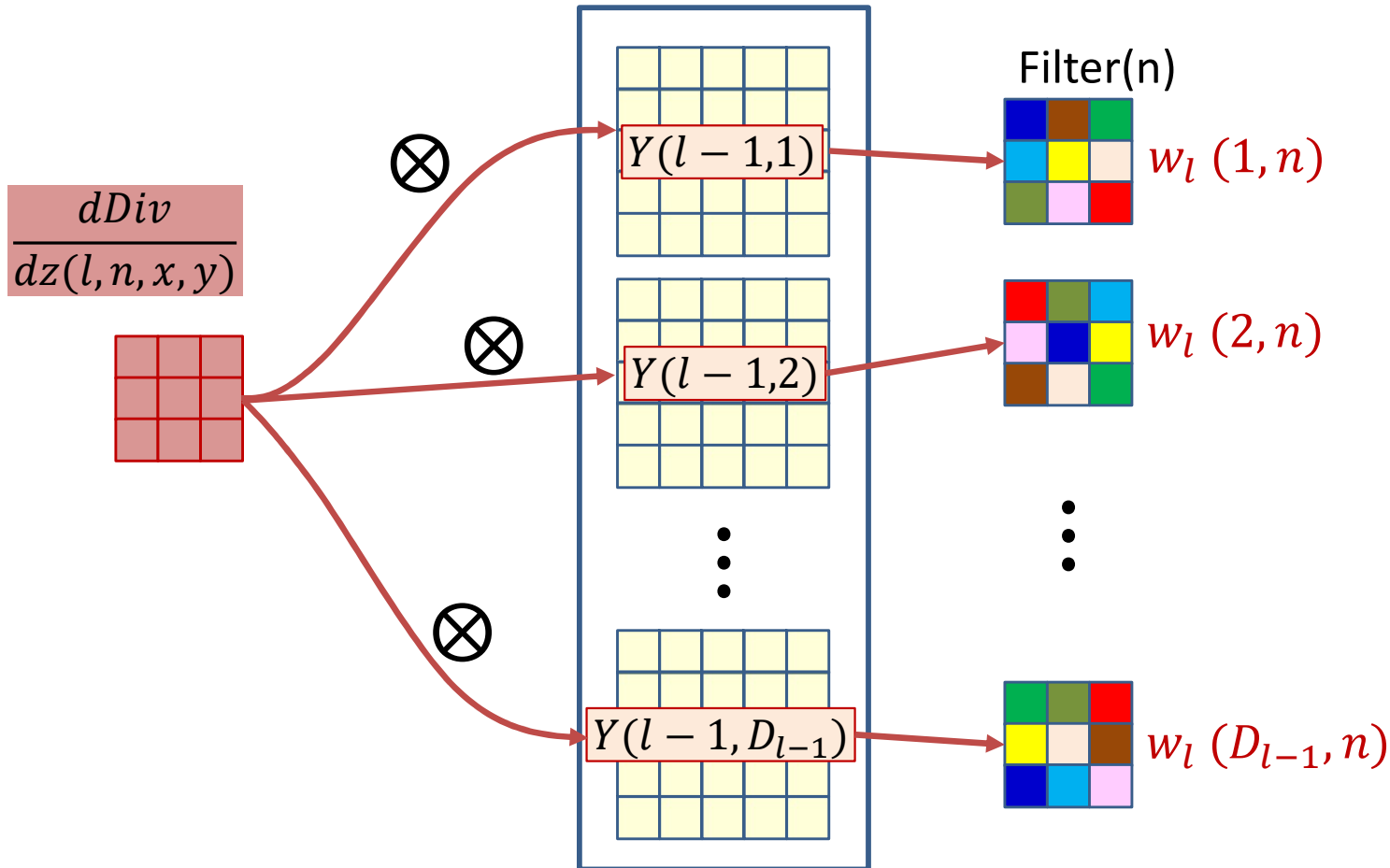
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The filter derivative



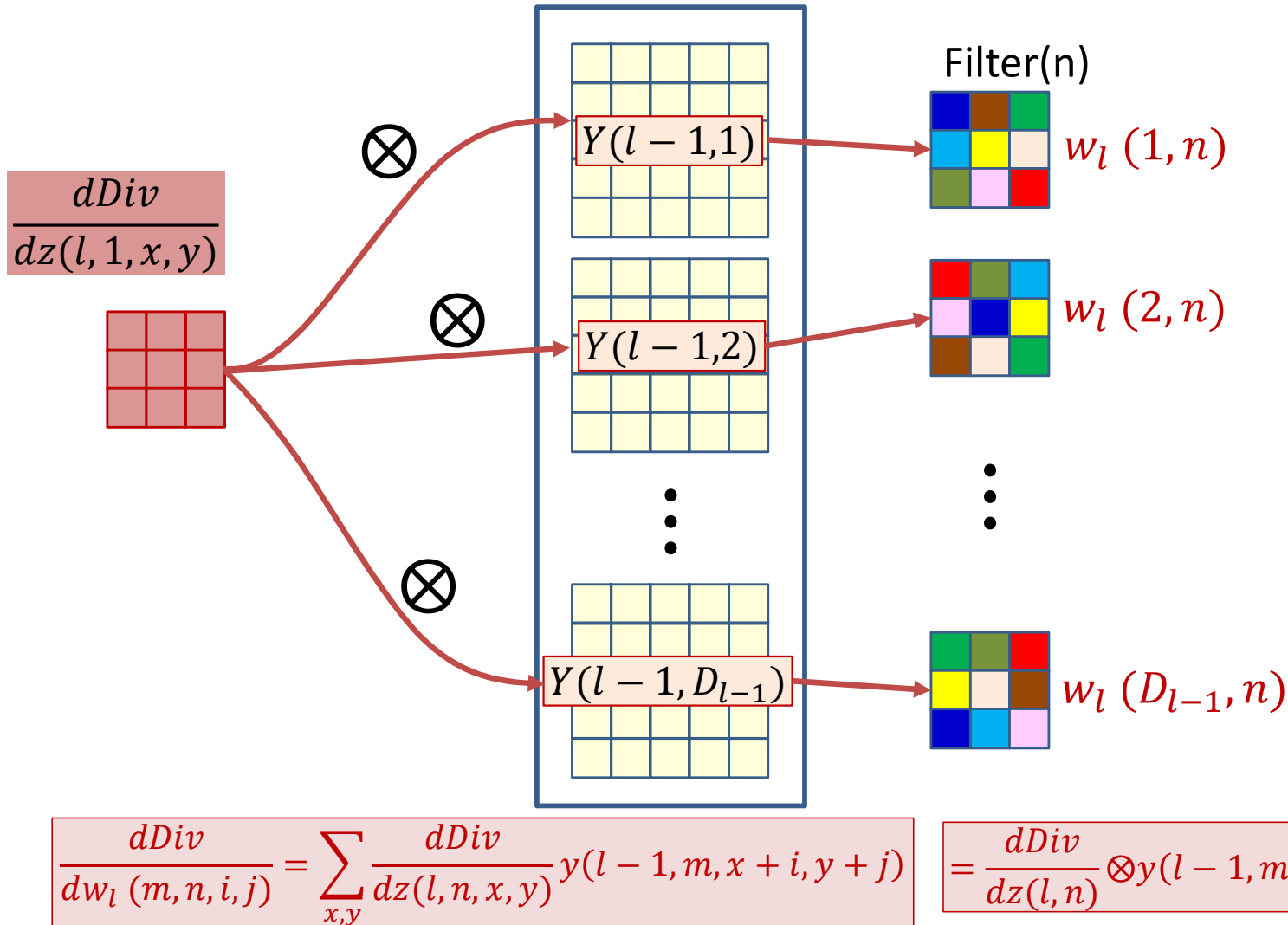
- The derivatives of the divergence w.r.t. every element of $Z(l, n)$ is known
 - Must use them to compute the derivative for $w_l(m, n, *, *)$

The filter derivative



- The derivative of the n^{th} affine map $Z(l, n)$ convolves with every output map $Y(l-1, m)$ of the $(l-1)^{\text{th}}$ layer, to get the derivative for $w_l(m, n)$, the m^{th} “channel” of the n^{th} filter

The filter derivative



If $Y(l-1,m)$ was zero padded in the forward pass, it must be zero padded for backprop

Poll 4: @761

Select all statements that are true about how to compute the derivative of the divergence w.r.t lth layer filters using backpropagation

- The derivative for the mth plane of the nth filter is computed by convolving the mth input (l-1th) layer map with the nth output (lth) layer affine derivative map
- The output map must be flipped left-right/up-down before convolution

Poll 4

Select all statements that are true about how to compute the derivative of the divergence w.r.t l th layer filters using backpropagation

- **The derivative for the m th plane of the n th filter is computed by convolving the m th input ($l-1$ th) layer map with the n th output (l th) layer affine derivative map**
- The output map must be flipped left-right/up-down before convolution

Derivatives for the filters at layer l :

Vector notation

```
# The weight  $W(l, j)$  is a 3D  $D_{l-1} \times K_1 \times K_1$   
# Assuming that derivative maps have been upsampled  
#   if stride > 1  
# Also assuming  $y$  map has been zero-padded if this was  
#   also done in the forward pass  
# The width and height of the  $dz$  map are  $W$  and  $H$ 
```

```
for n = 1:Dl  
  for x = 1:K1  
    for y = 1:K1  
      for m = 1:Dl-1  
        dw(l, m, n, x, y) = dz(l, n, :, :).           #dot product  
                           y(l-1, m, x:x+H-1, y:y+W-1)
```

Derivatives through a convolutional layer

- The entire process is simpler if we simply look at it through code
 - Through the reapplication of two simple rules:

- For any computation of the form

$$y = \sigma(z)$$

- The loss derivative for z given the loss derivative of y is

$$\frac{dL}{dz} = \frac{dL}{dy} \sigma'(z)$$

- For any computation in the forward pass

$$z = wy$$

- The backward computation to compute loss derivatives for the terms on the right, given loss derivatives to the left is

$$dL/dy += w dL/dz ; dL/dw += y dL/dz$$

- Since this is “backpropagation”, all computations are reversed

CNN: Forward

```
Y(0, :, :, :) = Image
for l = 1:L # layers operate on vector at (x,y)
    for x = 1:Wl-1-Kl+1
        for y = 1:Hl-1-Kl+1
            for j = 1:Dl
                z(l, j, x, y) = 0
                for i = 1:Dl-1
                    for x' = 1:Kl
                        for y' = 1:Kl
                            z(l, j, x, y) += w(l, j, i, x', y')
                                Y(l-1, i, x+x'-1, y+y'-1)
                Y(l, j, x, y) = activation(z(l, j, x, y))
Y = softmax( Y(L, :, 1, 1) .. Y(L, :, W-K+1, H-K+1) )
```

Switching to 1-based indexing with appropriate adjustments

Backward layer l

```
dw(l) = zeros(Dl×Dl-1×Kl×Kl)
dY(l-1) = zeros(Dl-1×Wl-1×Hl-1)
for x = Wl-1-Kl+1:downto:1
    for y = Hl-1-Kl+1:downto:1
        for j = Dl:downto:1
            dz(l,j,x,y) = dY(l,j,x,y) . f'(z(l,j,x,y))
            for i = Dl-1:downto:1
                for x' = Kl:downto:1
                    for y' = Kl:downto:1
                        dY(l-1,i,x+x'-1,y+y'-1) +=
                            w(l,j,i,x',y') dz(l,j,x,y)
                        dw(l,j,i,x',y') +=
                            dz(l,j,x,y) Y(l-1,i,x+x'-1,y+y'-1)
```

Complete Backward (no pooling)

```
dY(L) = dDiv/dY(L)
for l = L:downto:1    # Backward through layers
    dw(l) = zeros(Dl×Dl-1×Kl×Kl)
    dY(l-1) = zeros(Dl-1×Wl-1×Hl-1)
    for x = Wl-1-Kl+1:downto:1
        for y = Hl-1-Kl+1:downto:1
            for j = Dl:downto:1
                dz(l,j,x,y) = dY(l,j,x,y).f'(z(l,j,x,y))
                for i = Dl-1:downto:1
                    for x' = Kl:downto:1
                        for y' = Kl:downto:1
                            dY(l-1,i,x+x'-1,y+y'-1) +=
                                w(l,j,i,x',y')dz(l,j,x,y)
                            dw(l,j,i,x',y') +=
                                dz(l,j,x,y)y(l-1,i,x+x'-1,y+y'-1)
```

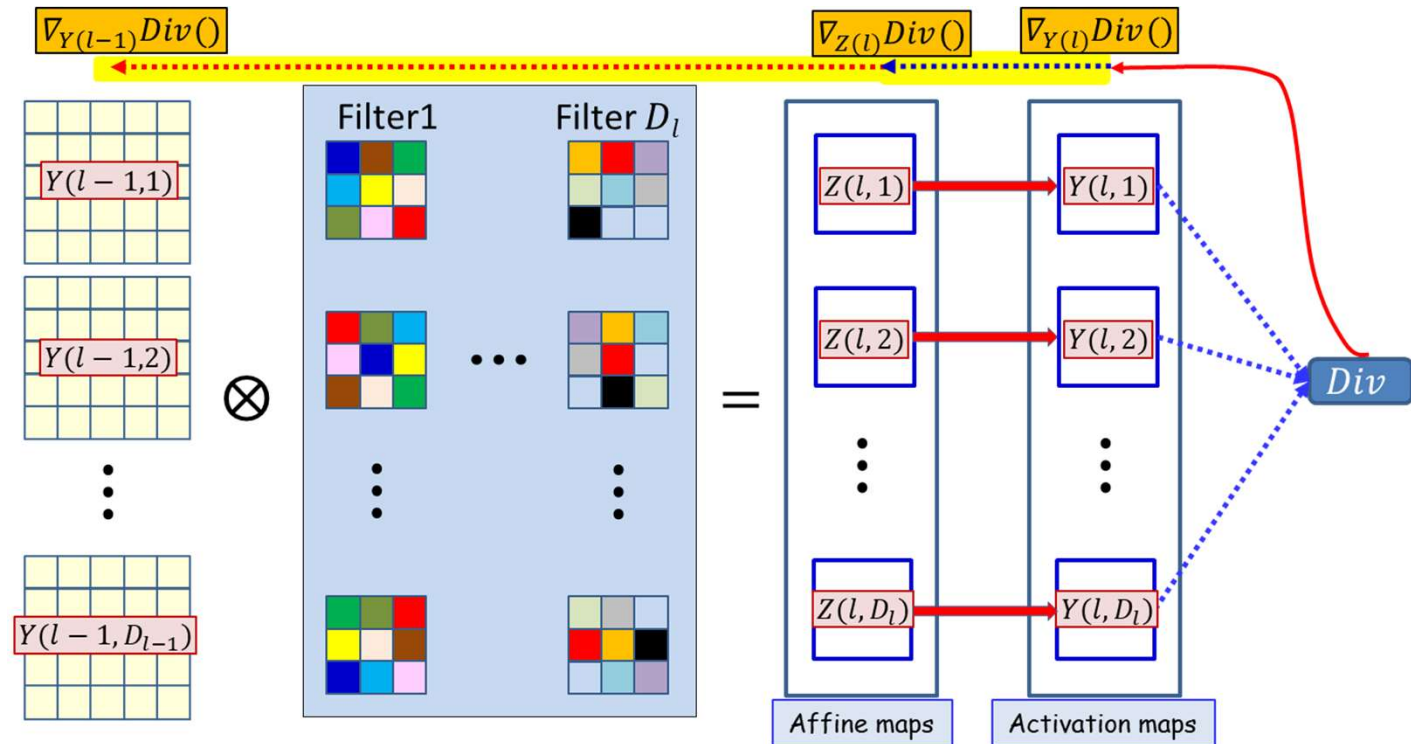
Complete Backward (no pooling)

```
dY(L) = dDiv/dY(L)
for l = L:downto:1    # Backward through layers
    dw(l) = zeros(Dl×Dl-1×Kl×Kl)
    dY(l-1) = zeros(Dl-1×Wl-1×Hl-1)
    for x = Wl-1-Kl+1:downto:1
        for y = Hl-1-Kl+1:downto:1
            for j = Dl:downto:1
                dz(l,j,x,y) = dY(l,j,x,y) . f'(z(l,j,x,y))
                for i = Dl-1:downto:1
                    for x' = Kl:downto:1
                        for y' = Kl:downto:1
                            dY(l-1,i,x+x'-1,y+y'-1) +=
                                w(l,j,i,x',y') dz(l,j,x,y)
                            dw(l,j,i,x',y') +=
                                dz(l,j,x,y) y(l-1,i,x+x'-1,y+y'-1)
```

Multiple ways of recasting this as tensor/ vector operations.

Will not discuss here

Backpropagation: Convolutional layers



- **For convolutional layers:**



How to compute the derivatives w.r.t. the affine $Z(l)$ maps from the activation output maps $Y(l)$



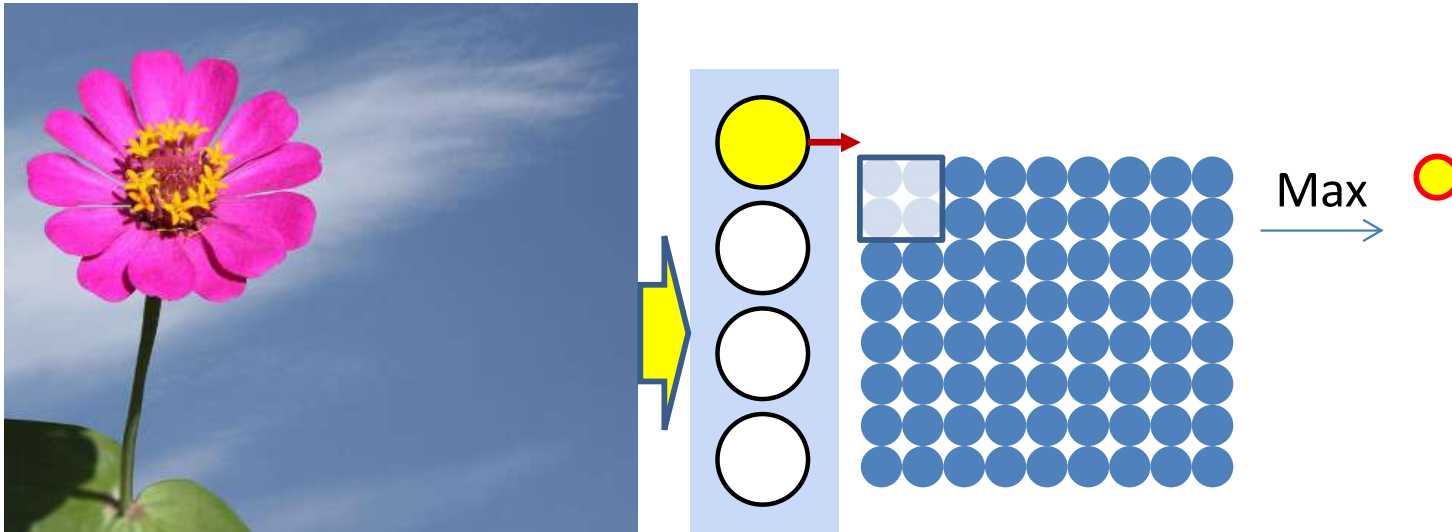
How to compute the derivative w.r.t. $Y(l-1)$ and $w(l)$ given derivatives w.r.t. $Z(l)$

Backpropagation: Convolutional and Pooling layers

- **Assumption:** We already have the derivatives w.r.t. the elements of the maps output by the final convolutional (or pooling) layer
 - Obtained as a result of backpropagating through the flat MLP
- **Required:**
 - **For convolutional layers:**
 - How to compute the derivatives w.r.t. the affine $Z(l)$ maps from the activation output maps $Y(l)$
 - How to compute the derivative w.r.t. $Y(l - 1)$ and $w(l)$ given derivatives w.r.t. $Z(l)$
 - **For pooling layers:**
 - How to compute the derivative w.r.t. $Y(l - 1)$ given derivatives w.r.t. $Y(l)$

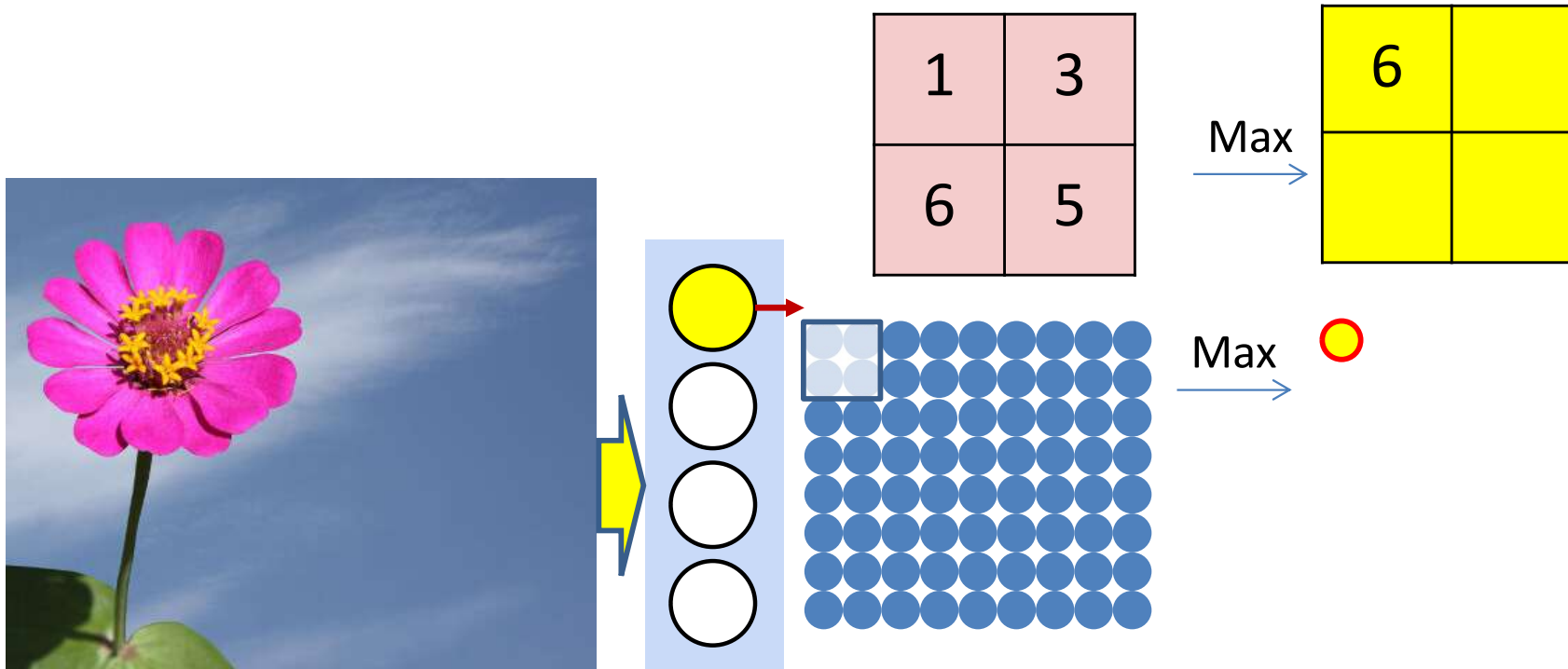


Pooling



- Pooling “pools” groups of values to reduce jitter-sensitivity
 - Scanning with a “pooling” filter
- The most common pooling is “Max” pooling

Max Pooling

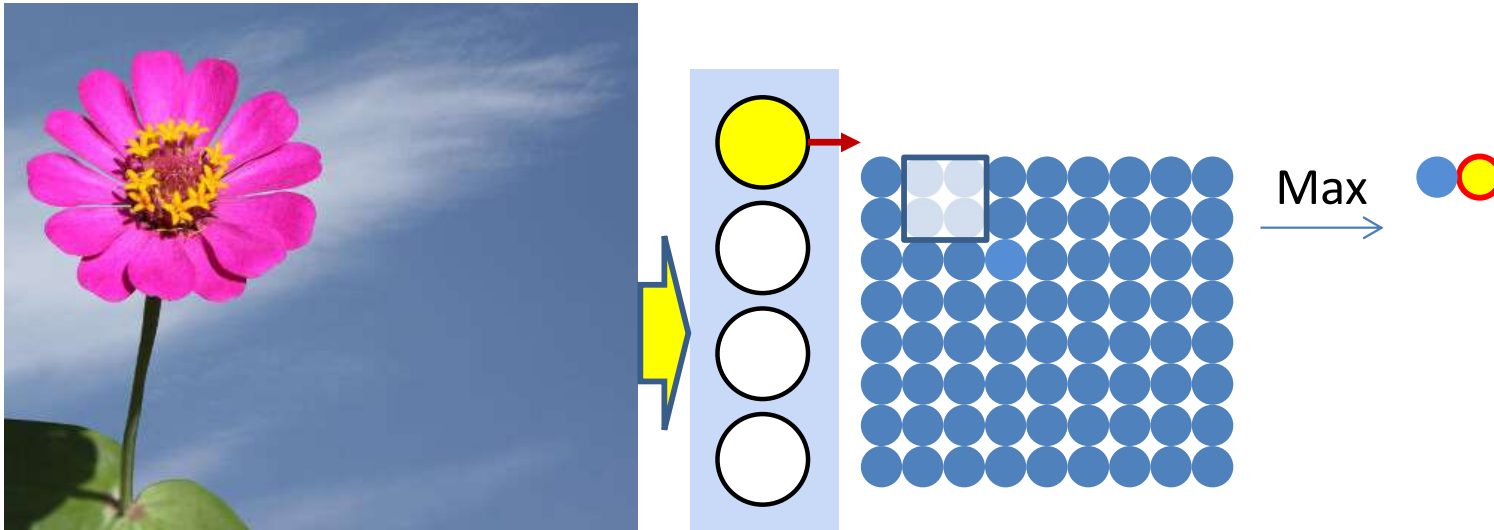


- Max pooling selects the largest from a pool of elements
- Pooling is performed by “scanning” the input

$$P(l, m, i, j) = \underset{\substack{k \in \{i, i+K_{lpool}-1\}, \\ n \in \{j, j+K_{lpool}-1\}}}{\text{argmax}} Y(l-1, m, k, n)$$

$$Y(l, m, i, j) = Y(l-1, m, P(l, m, i, j))$$

Max pooling

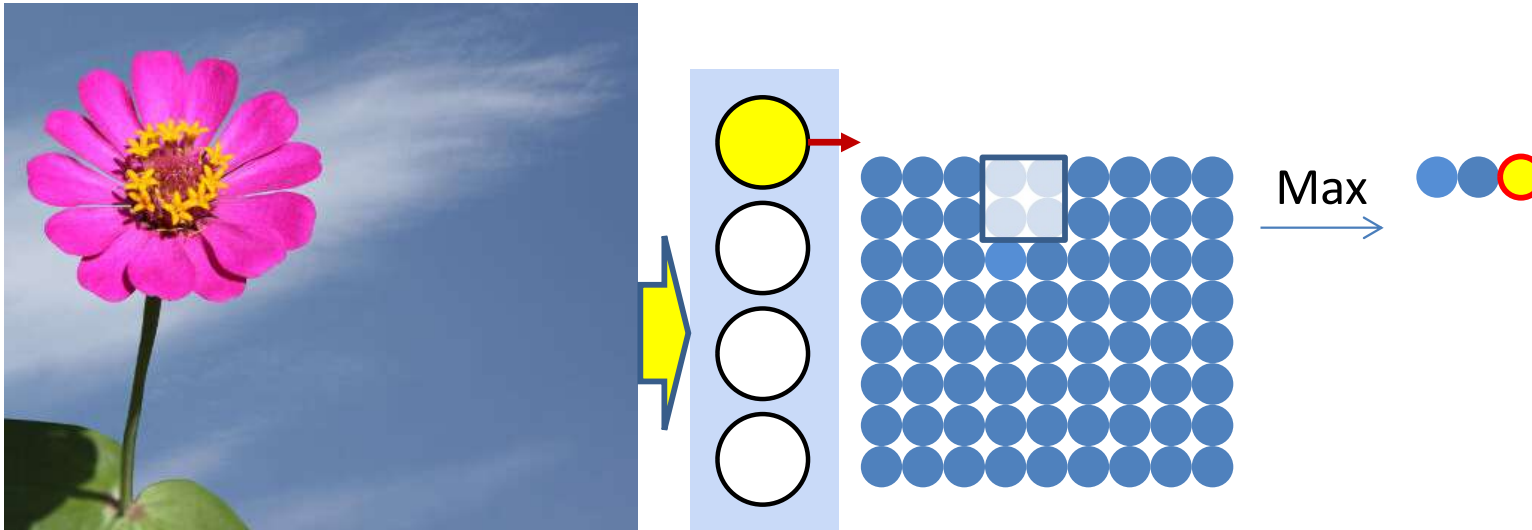


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$$Y(l, m, i, j) = Y(l-1, m, P(l, m, i, j))$$

Max pooling

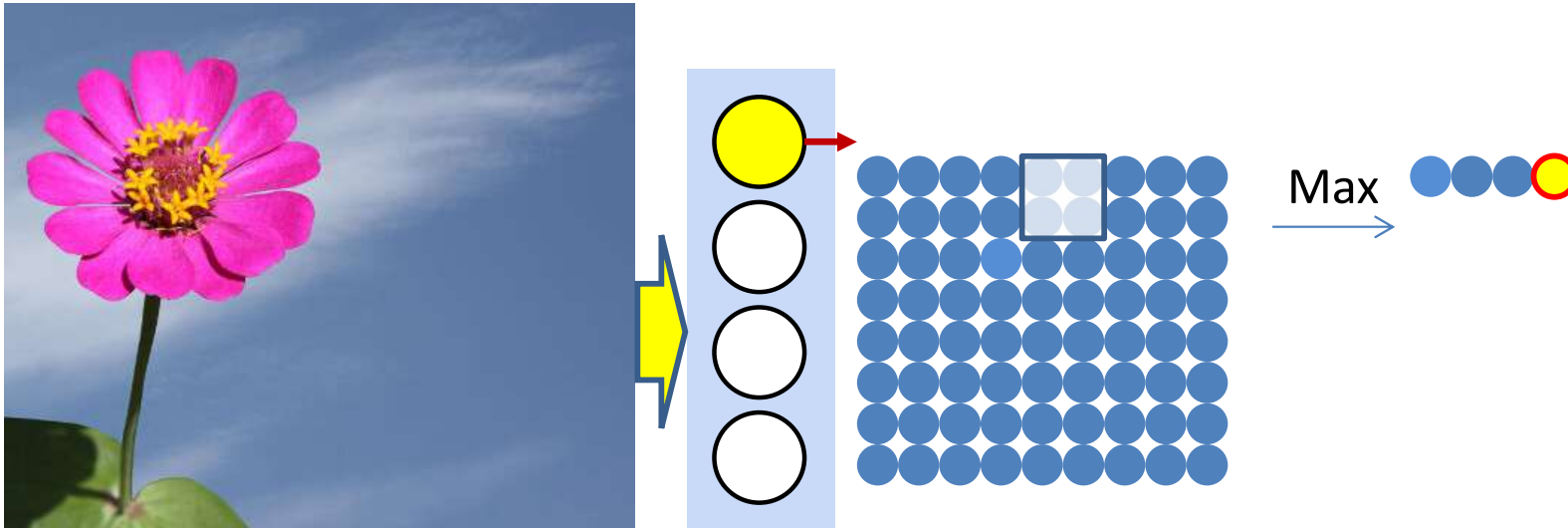


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$$Y(l, m, i, j) = Y(l-1, m, P(l, m, i, j))$$

Max pooling

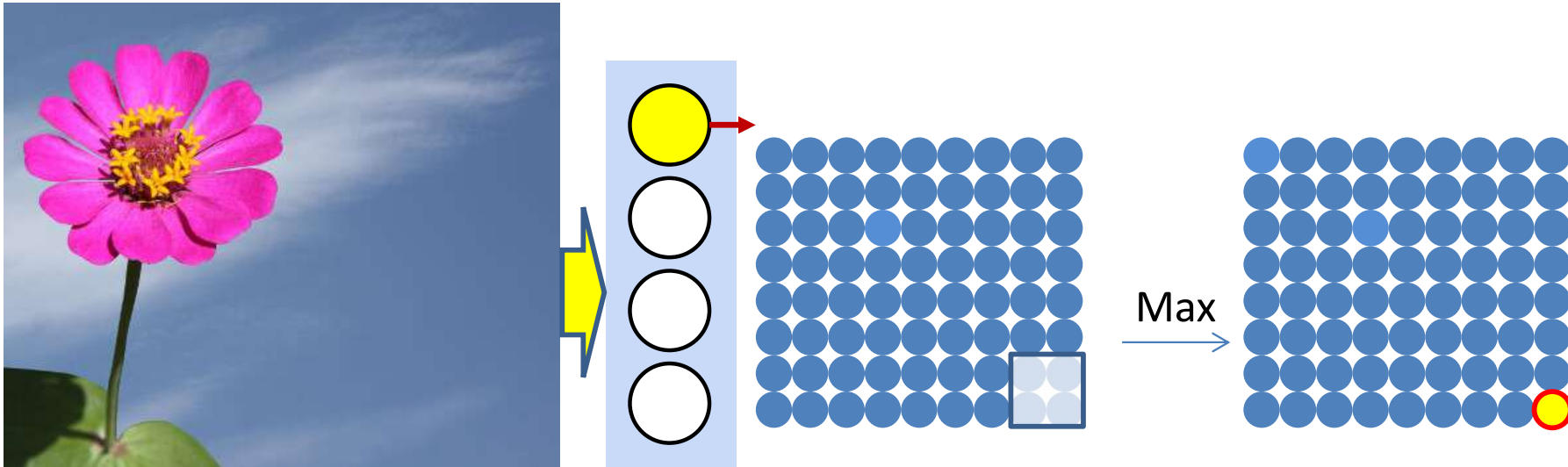


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$$Y(l, m, i, j) = Y(l-1, m, P(l, m, i, j))$$

Max pooling

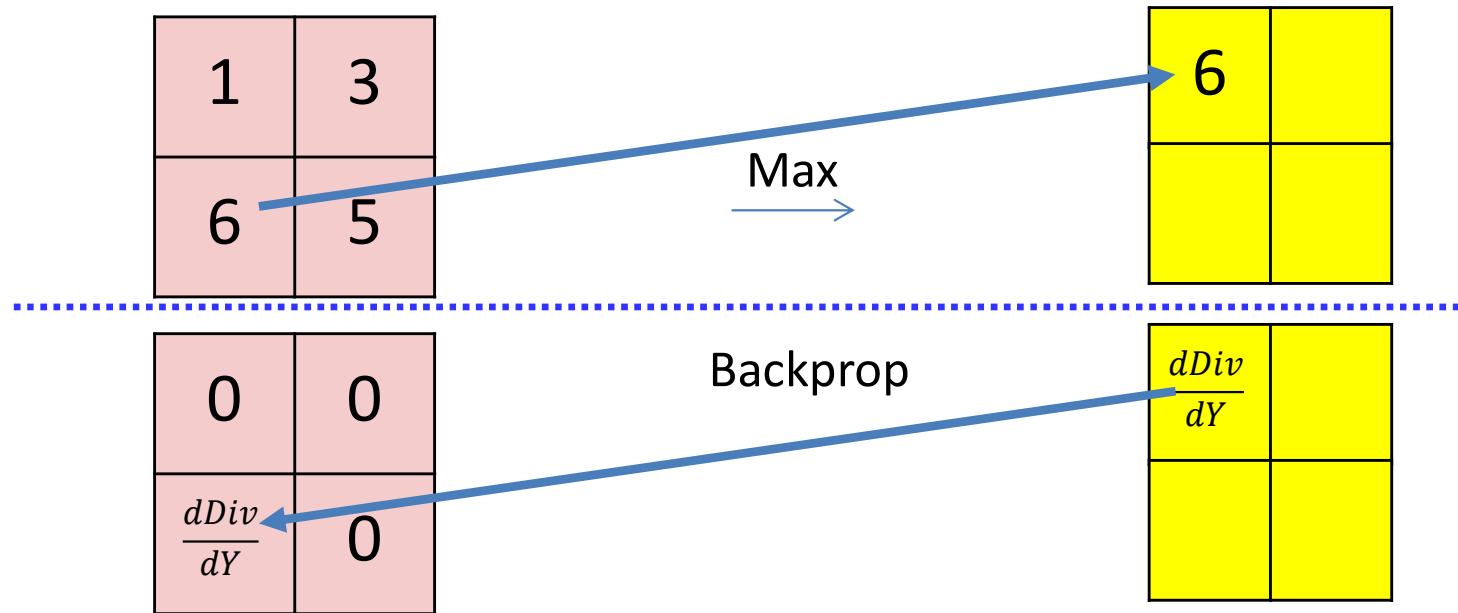


- Max pooling selects the largest from a pool of elements
- Pooling is performed by “scanning” the input

$$P(l, m, i, j) = \underset{\substack{k \in \{i, i+K_{lpool}-1\}, \\ n \in \{j, j+K_{lpool}-1\}}}{\text{argmax}} Y(l-1, m, k, n)$$

$$Y(l, m, i, j) = Y(l-1, m, P(l, m, i, j))$$

Derivative of Max pooling



$$\frac{dDiv}{dy(l-1, m, k, l)} = \begin{cases} \frac{dDiv}{dy(l, m, i, j)} & \text{if } (k, l) = P(l, m, i, j) \\ 0 & \text{otherwise} \end{cases}$$

- Max pooling selects the largest from a pool of elements

$$P(l, m, i, j) = \underset{\substack{k \in \{i, i+K_{lpool}-1\}, \\ n \in \{j, j+K_{lpool}-1\}}}{\text{argmax}} Y(l-1, m, k, n)$$

$$y(l, m, i, j) = y(l-1, m, P(l, m, i, j))$$

Max Pooling layer at layer l

- a) Performed separately for every map (j).
 - *) Not combining multiple maps within a single max operation.
- b) Keeping track of location of max

Max pooling

```
for j = 1:D1
    for x = 1:Wl-1-K1+1
        for y = 1:Hl-1-K1+1
            pidx(l,j,x,y) = maxidx(y(l-1,j,x:x+K1-1,y:y+K1-1))
            y(l,j,x,y) = y(l-1,j,pidx(l,j,x,y))
```



Derivative of max pooling layer at layer l

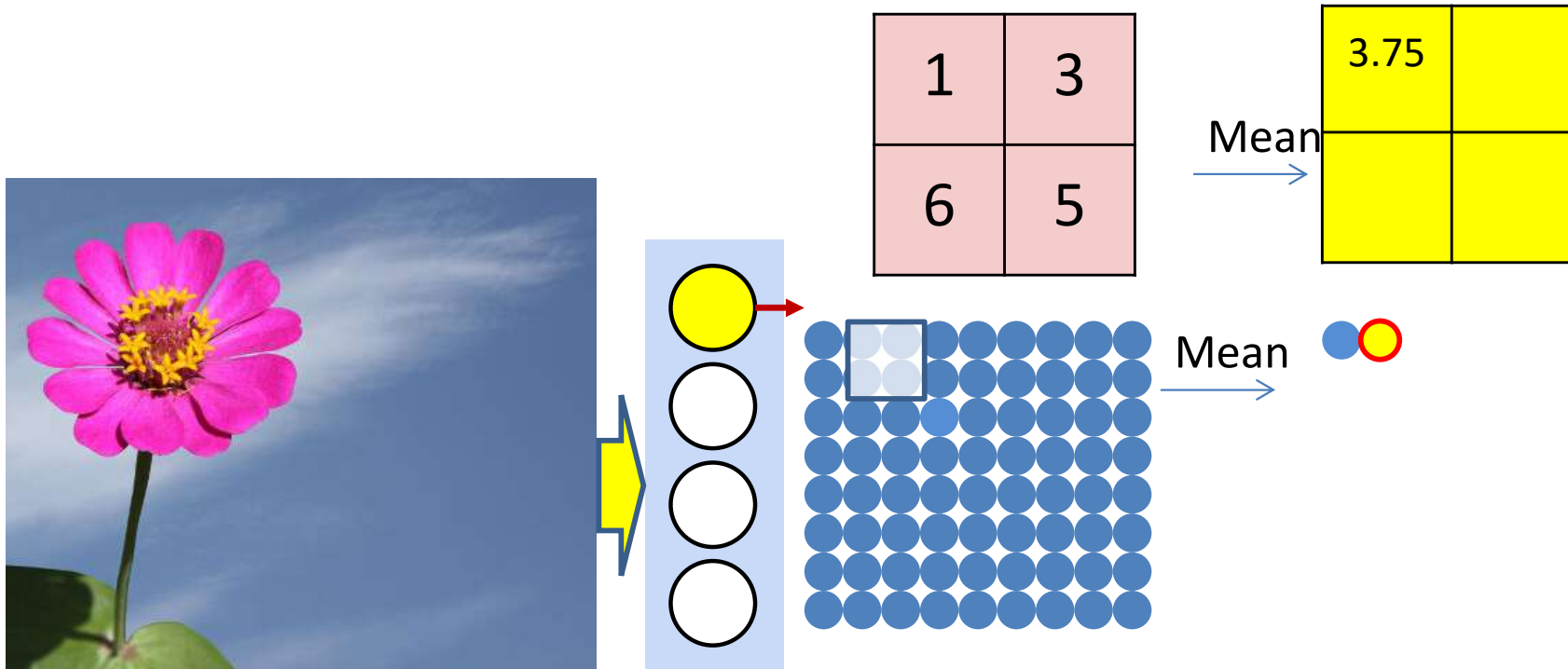
- a) Performed separately for every map (j).
 - *) Not combining multiple maps within a single max operation.
 - b) Keeping track of location of max
- 

Max pooling

```
dy(:, :, :) = zeros(D1 x W1 x H1)
for j = 1:D1
    for x = 1:W1
        for y = 1:H1
            dy(l-1, j, pidx(l, j, x, y)) += dy(l, j, x, y)
```

“+=” because this entry may be selected in multiple adjacent overlapping windows

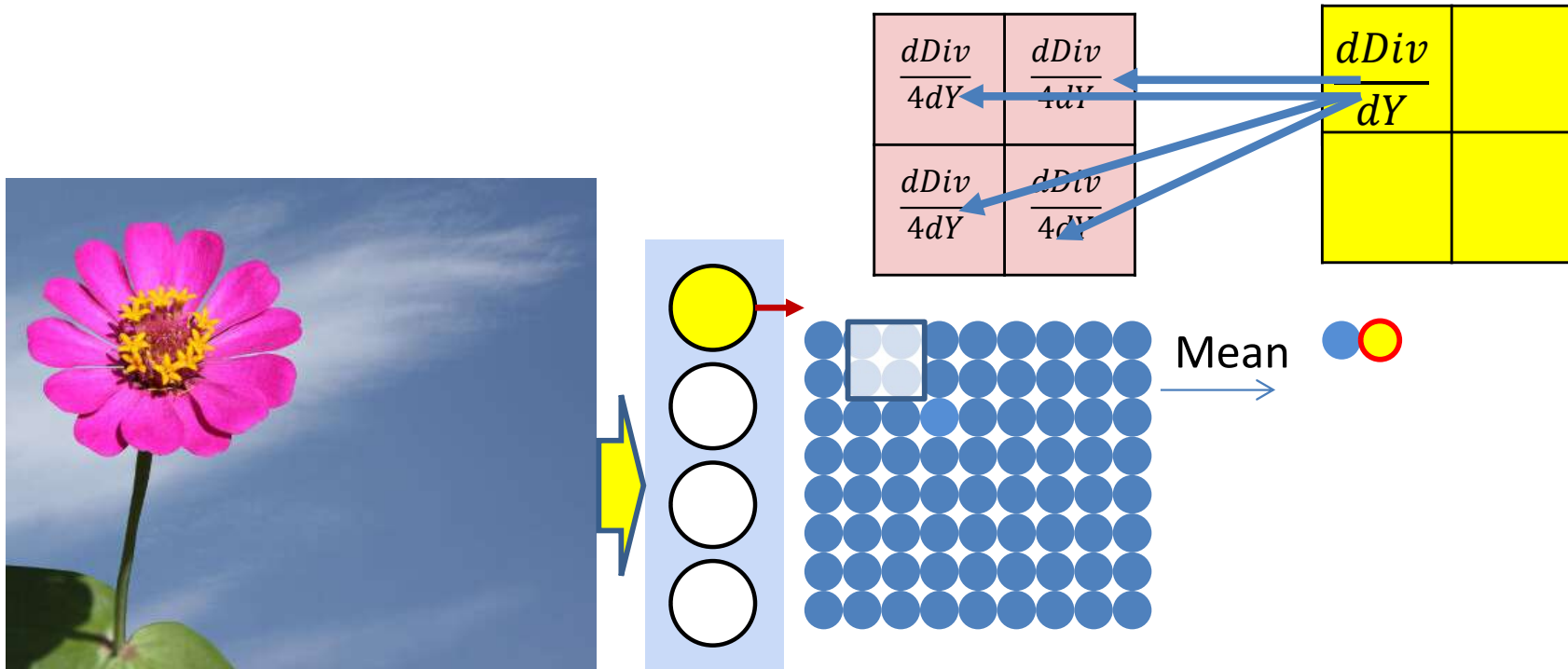
Mean pooling



- Mean pooling compute the mean of a pool of elements
- Pooling is performed by “scanning” the input

$$y(l, m, i, j) = \frac{1}{K_{lpool}^2} \sum_{\substack{k \in \{i, i+K_{lpool}-1\}, \\ n \in \{j, j+K_{lpool}-1\}}} y(l-1, m, k, n)$$

Derivative of mean pooling



- The derivative of mean pooling is distributed over the pool

$$\begin{aligned}
 & k \in \{i, i + K_{lpool} - 1\}, \\
 & n \in \{j, j + K_{lpool} - 1\}
 \end{aligned}
 \quad
 dy(l-1, m, k, n) += \frac{1}{K_{lpool}^2} dy(l, m, k, n)$$

Mean Pooling layer at layer l

Mean pooling

```
for j = 1:Dl #Over the maps
    for x = 1:Wl-1-Kl+1 #Kl = pooling kernel size
        for y = 1:Hl-1-Kl+1
            y(l,j,x,y) = mean(y(l-1,j,x:x+Kl-1,y:y+Kl-1))
```

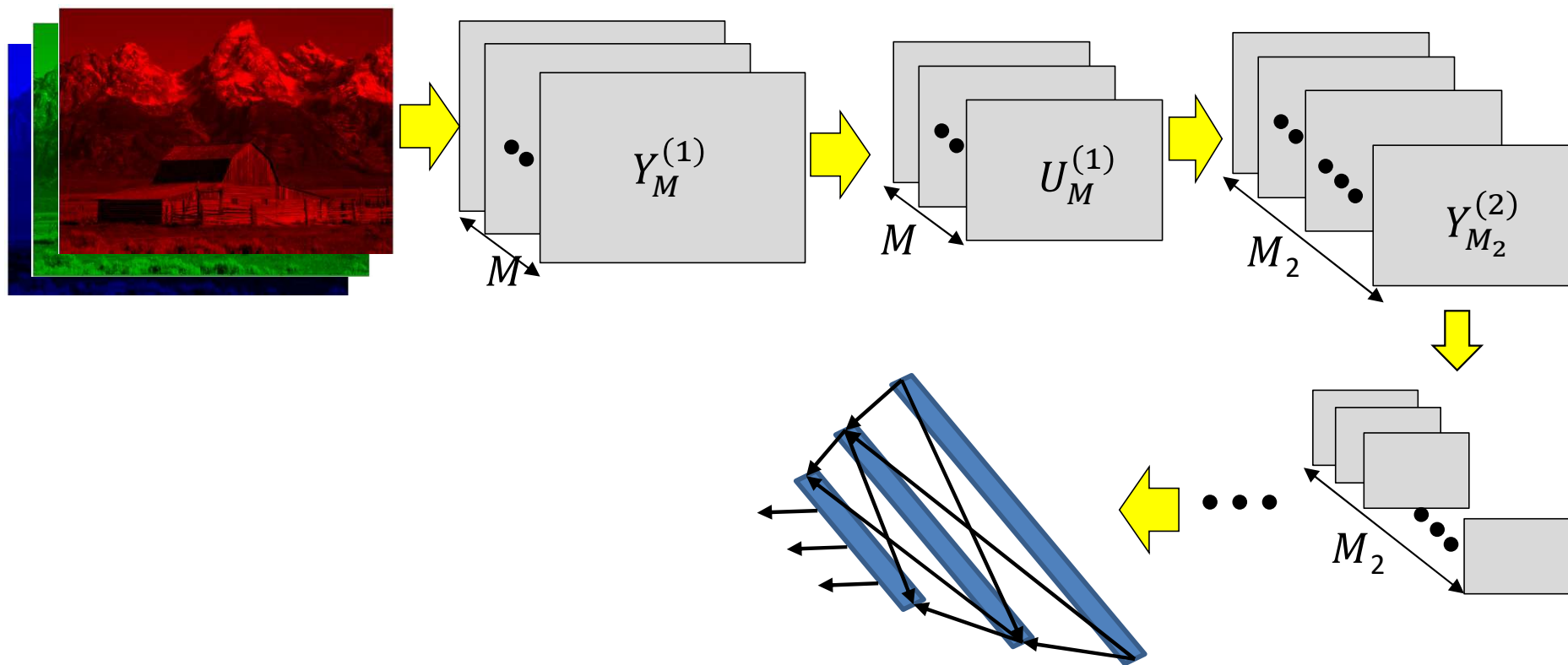
Derivative of mean pooling layer at layer l

Mean pooling

```
dy(:, :, :) = zeros(D1 x W1 x H1)
for k = 1:D1
    for x = 1:W1
        for y = 1:H1
            for i = 1:K1pool
                for j = 1:K1pool
                    dy(l-1, k, p, x+i, y+j) += (1/K1pool2) dy(l, k, x, y)
```

“+=” because adjacent windows may overlap

Learning the network



- Have shown the derivative of divergence w.r.t every intermediate output, and every free parameter (filter weights)
- Can now be embedded in gradient descent framework to learn the network
- Still missing one component... resampling
 - Next class

Story so far

- The CNN is a supervised version of a computational model of mammalian vision
- It includes
 - Convolutional layers comprising learned filters that scan the outputs of the previous layer, followed by an activation function
 - Pooling layers that operate over groups of outputs from the convolutional layer for jitter invariance
 - Optional resizing operations, typically performed through strides greater than 1 (for downsampling) or interpolation of zeros (for upsampling)
- The parameters of the network can be learned through regular back propagation
 - Computing derivatives for convolutional layers is simply also a convolution operation
 - Convolve transposed filters with derivative maps of output for derivatives of input maps
 - Convolve input maps with derivative maps of the output for derivatives of filters
 - Backprop of pooling operations redistributes the derivatives at the output over all the inputs that actively contributed to it