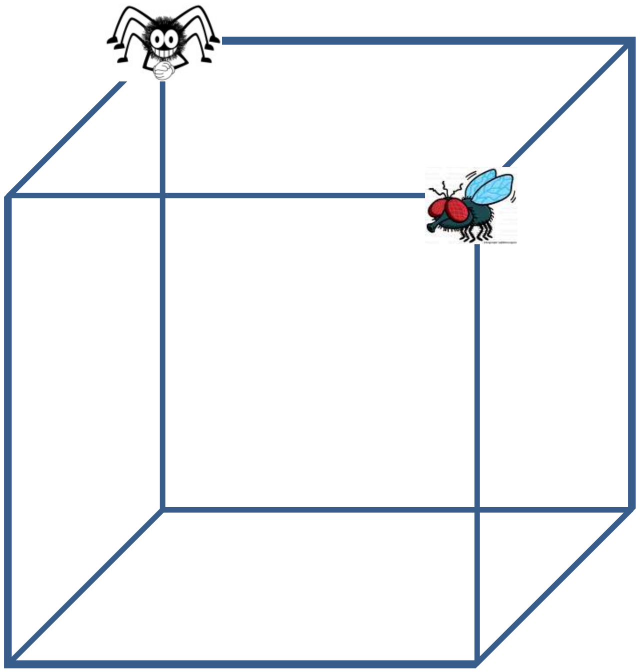
Reinforcement Learning

11-785, Spring 2020

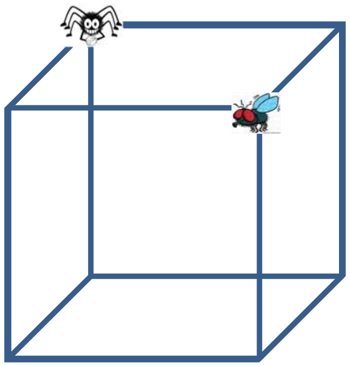
Defining MDPs, Planning

The story of Flider and Spy



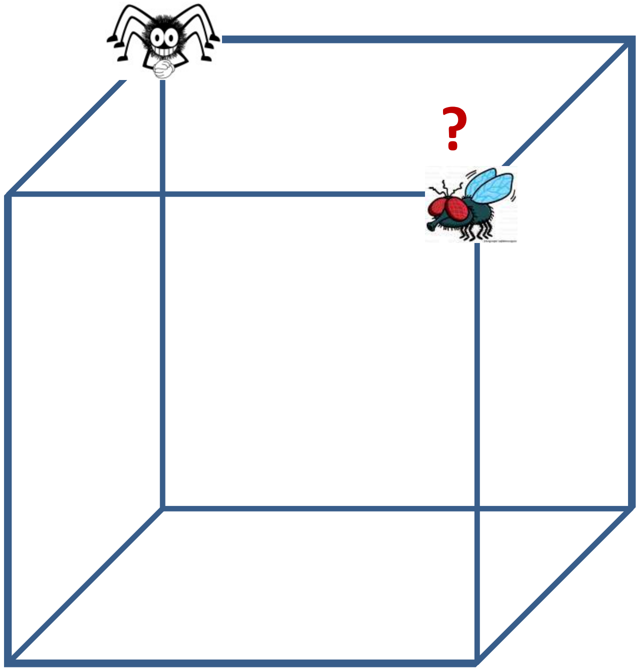
- Flider the spider is at the far corner of the room, and Spy the fly is sleeping happily at the near corner

The story of Flider and Spy



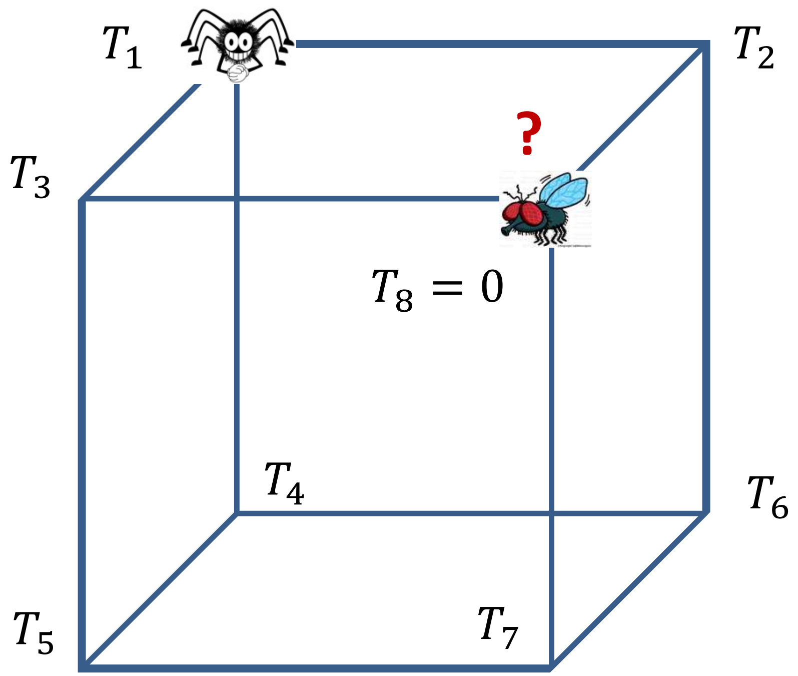
- Flider only walks along edges
- She begins walking along one of the three edges at random
- She takes one minute to cover the distance from one corner to the other along any edge
- When she arrives at the new corner, she randomly chooses one of the three edges and continues walking (she may even turn back)

The story of Flider and Spy



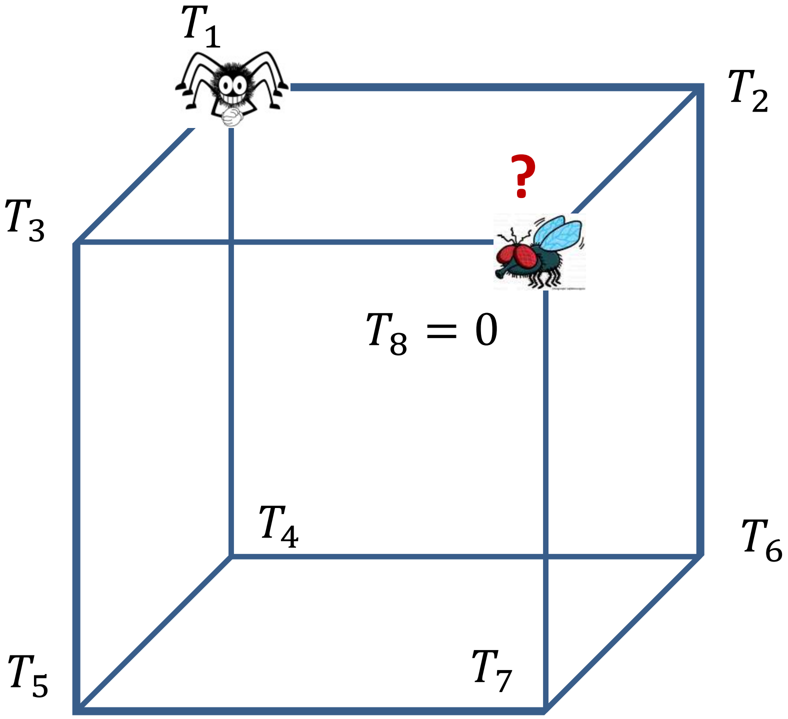
- What is the life expectancy of Spy?

Flider and Spy



- Let T_i be the life expectancy if Flider is at the i^th^ corner

Flider and Spy



$$T_1 = \frac{1}{3}(1 + T_2) + \frac{1}{3}(1 + T_3) + \frac{1}{3}(1 + T_4)$$

$$T_2 = \frac{1}{3}(1 + T_1) + \frac{1}{3}(1 + T_6) + \frac{1}{3}(1 + T_8)$$

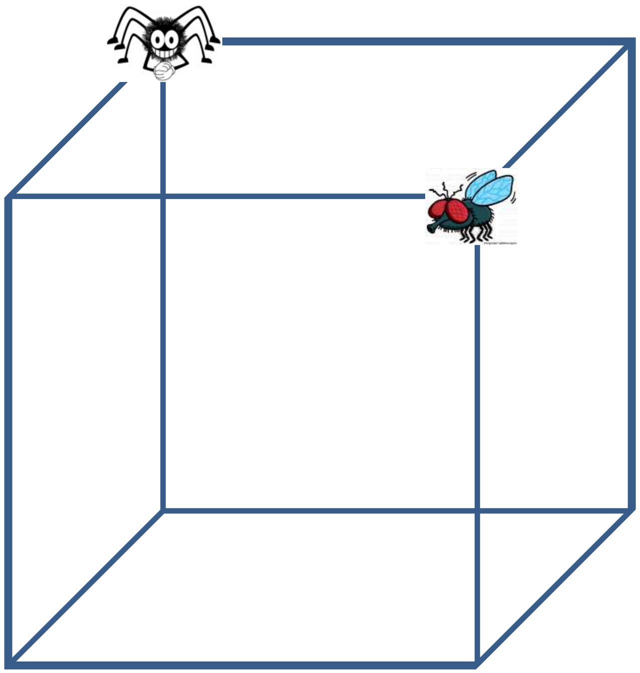
$$\vdots$$

$$T_7 = \frac{1}{3}(1 + T_8) + \frac{1}{3}(1 + T_6) + \frac{1}{3}(1 + T_5)$$

$$T_8 = 0$$

- $1 + T_i$ is the life expectancy if Flider the Spider begins walking towards the ith corner
  - – 1 minute to get to the corner plus the time taken to get from that corner to Spy the fly
- 8 Equations, 8 unknowns, trivial to solve

A little terminology



- Markov Process: Does not matter how you got here, only matters where you are

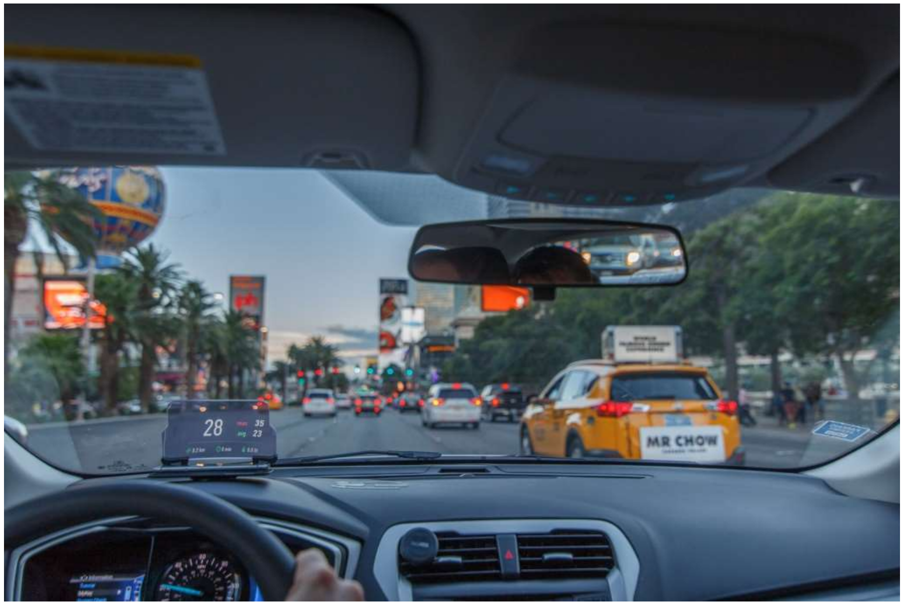
An interesting class of problems

- Is a move good?
  - – You will not know until the end of the game

An interesting class of problems

- Is an investment plan good?
  - You will not know for a while

An interesting class of problems



- Do I
  - Change lane left?
  - Change lane right?
  - Accelerate?
  - Decelerate?

Reward-based problems

- And many others

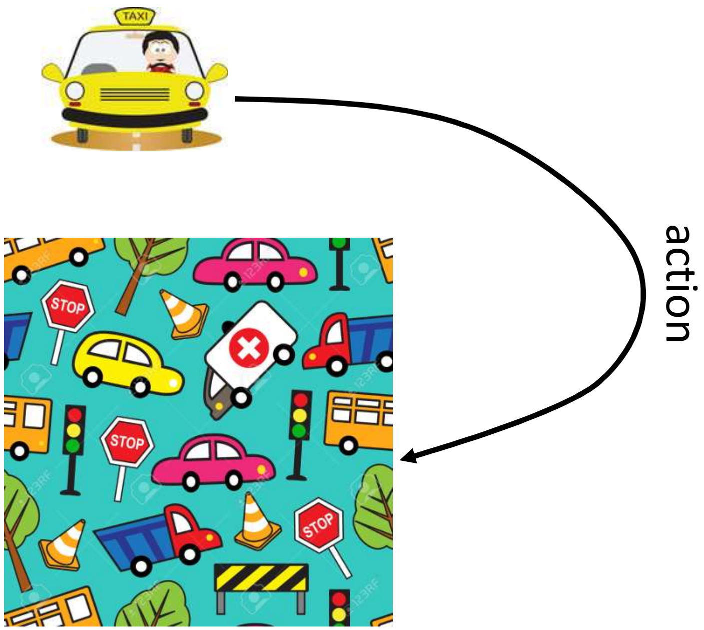
- Common theme: These are control problems where
  - Your actions beget rewards
    - Win the game
    - Make money
    - Get home sooner
  - But not deterministically
    - A world out there that is not predictable

- From experience of *belated* rewards, you must learn to act rationally

General cartoon of the world

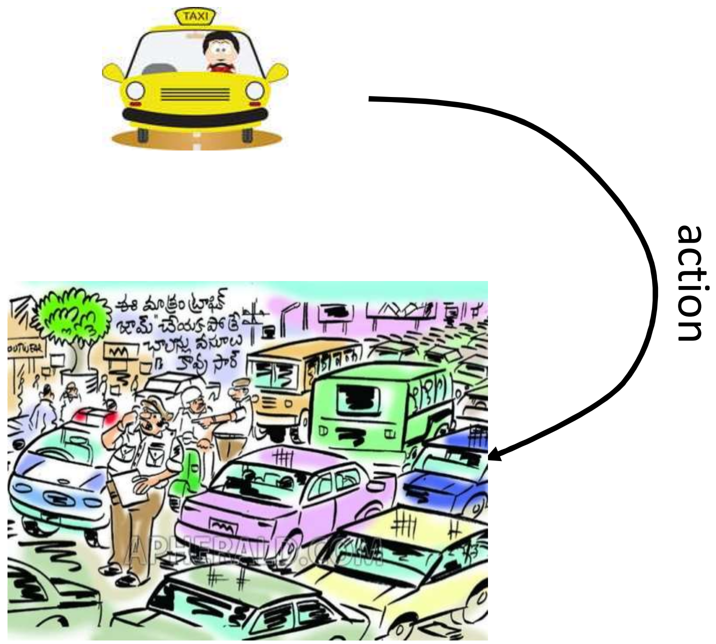
- Agent operates in an environment
  - Agent may be you..
  - Environment is the game, the market, the road..

General cartoon of the world



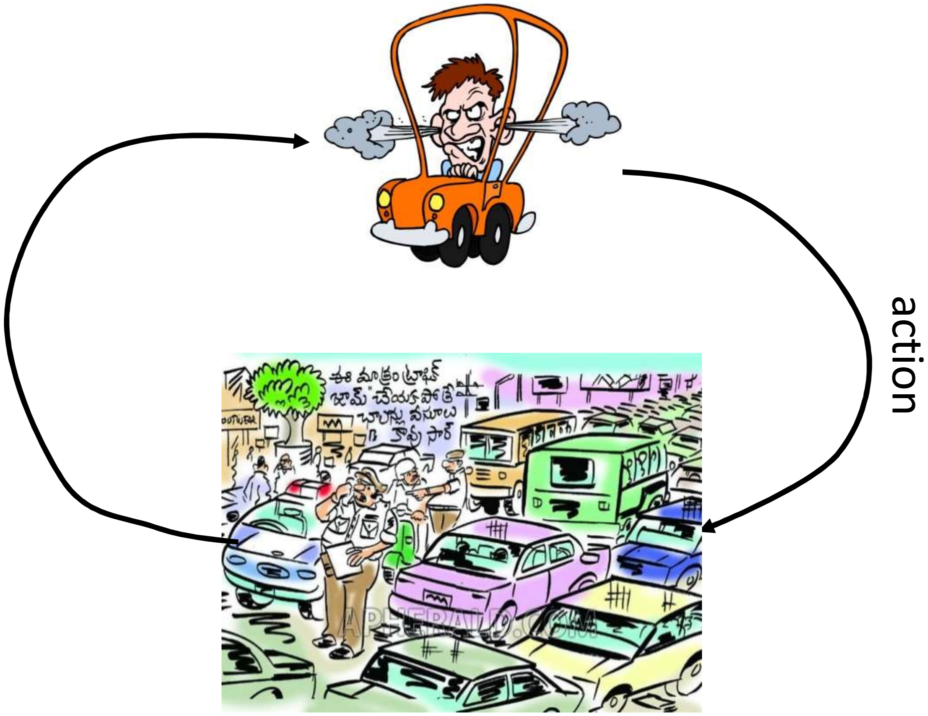
- Agent takes actions which affect the environment

General cartoon of the world



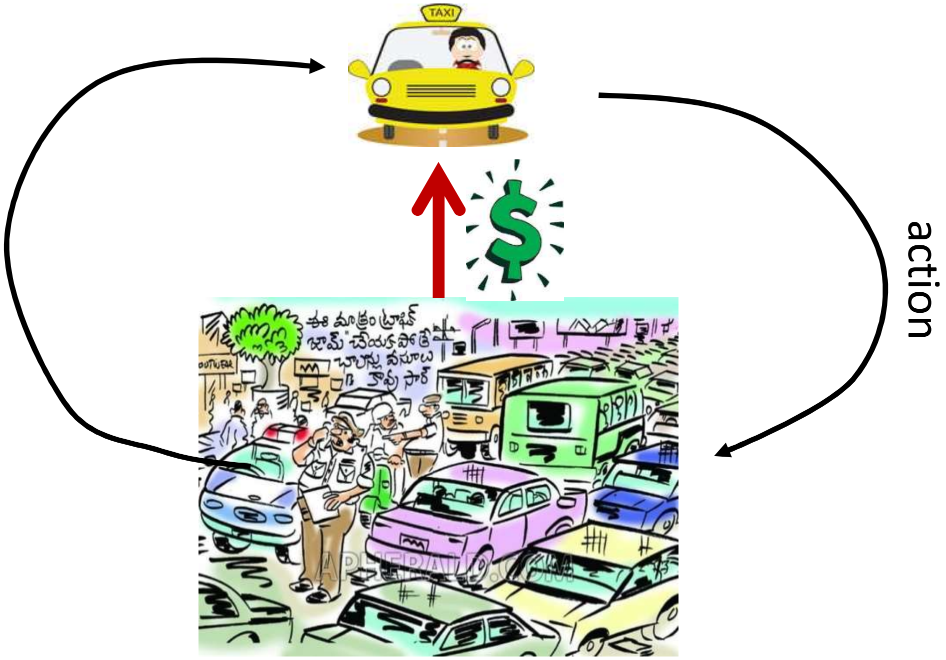
- Agent takes actions which affect the environment
- Which changes in a somewhat unpredictable way

General cartoon of the world



- Agent takes actions which affect the environment
- Which changes in a somewhat unpredictable way
- Which affects the agent's situation

General cartoon of the world



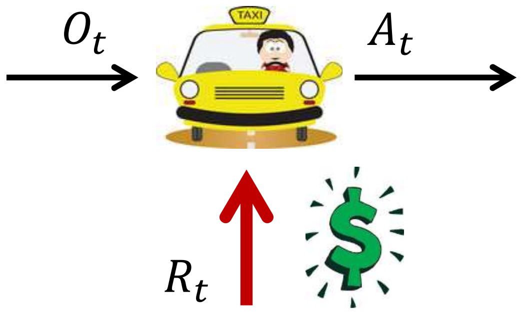
- The agent also receives rewards..
  - Which may be apparent immediately
  - Or not apparent for a very long time

Challenge

- How must the agent behave to maximize its rewards

Lets formalize the system

- At each time t the agent:
  - Makes an observation O_t of the environment
  - Receives a reward R_t
  - Performs an action A_t



From the perspective of the Agent

- What the agent perceives..
- The following History:
- $H_t = O_0, R_0, A_0, O_1, R_1, A_1, \dots, O_t, R_t$

- The total history at any time is the sequence of observations, rewards and actions
- We need to model this sequence such that at any time t, the best $A_t | H_t$ can be chosen
    - The Strategy that maximizes total reward $R_0 + R_1 + \cdots + R_T$

Lets formalize the system

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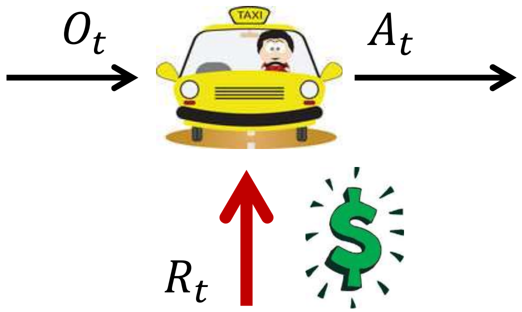
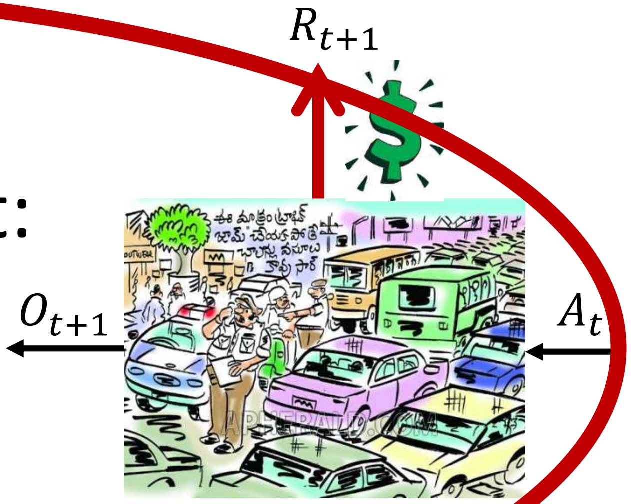


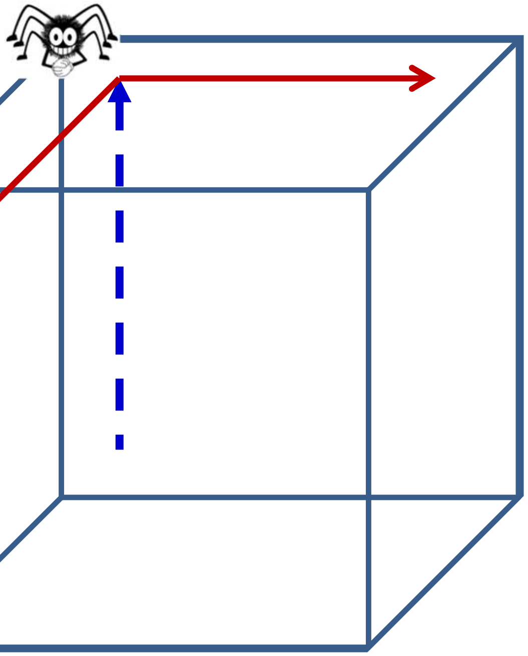
- At each time t the environment:
  - Receives an action A_t
  - Emits a reward R_{t+1}
  - Changes and produces an observation O_{t+1}



Can define an environment "state"

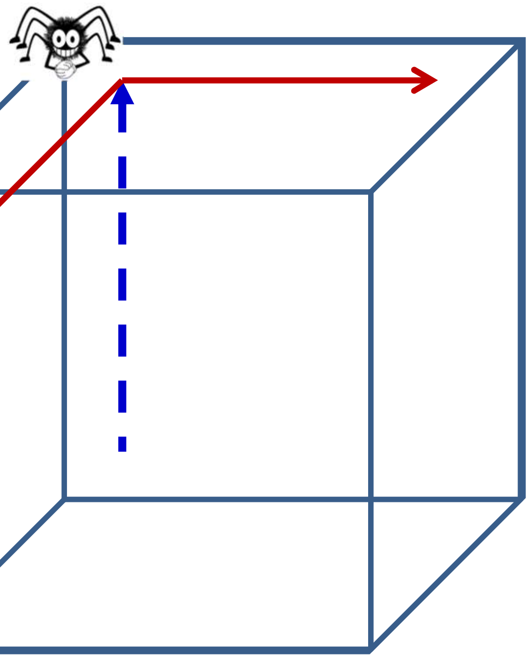
- Fully captures the "status" of the system
  - E.g., in an automobile: [position, velocity, acceleration]
  - In traffic: the position, velocity, acceleration of *every* vehicle on the road
  - In Chess: the state of the board + whose turn it is next

A brief trip to Nostalgia..



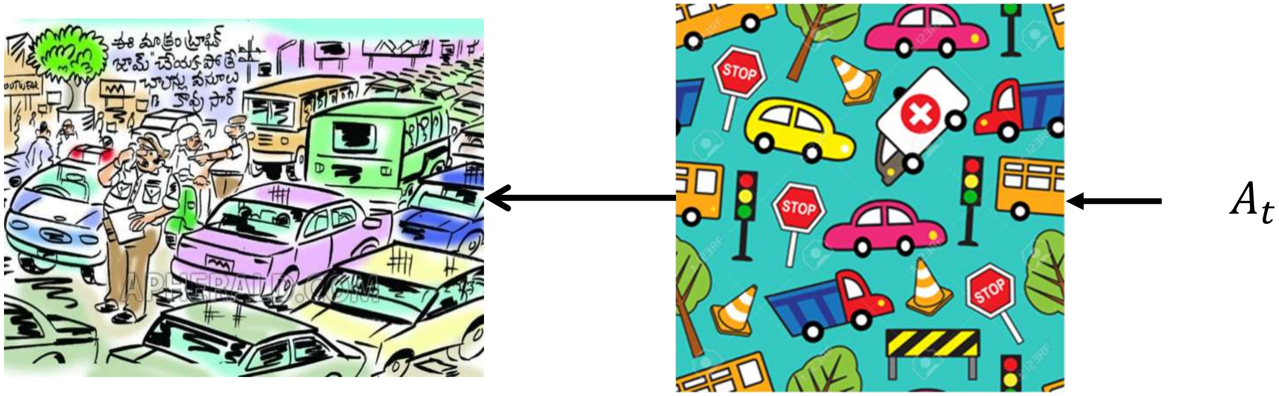
- Glider, Flider’s brother, never turns around during his wanderings
  - On arriving at any corner, he chooses one of the two “forward” paths randomly.
    - The future possibilities depend on the edge he arrived from
  - Is he Markovian?

Glider is a Markov dude!



- Any causal system can be viewed as Markov, with appropriately defined state
  - The *Information state* S_t may differ from the *apparent* state s_t
  - Defining $S_t = s_1, s_2, \ldots, s_t$
  - $P(S_{t+1} | S_0, S_1, \ldots, S_t) = P(S_{t+1} | S_t)$

Markov property



- **Assumption: The *information state* of the environment is Markov**

$$P(S_{t+1}|S_0, S_1, \dots, S_t) = P(S_{t+1}|S_t)$$

- The environment’s future only depends on its present

To Maximize Reward

- The agent must *model* this environment process
  - Formulate its own model for the environment, which must ideally match the true values as closely as possible
    - Based only on what it observes

- Agent must formulate winning strategy based on model of environment

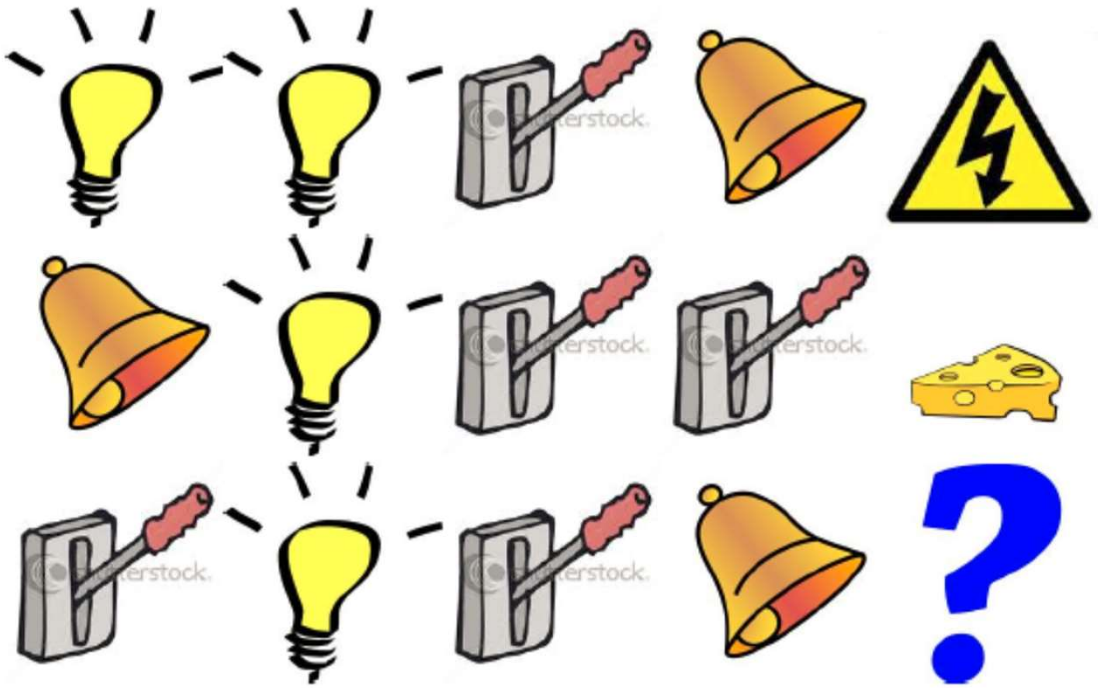
The Agent’s Side of the Story

- Agent has an internal representation of the environment state
  - May not match the true one at all

- May be defined in any manner
  - Formally the agent state $S_t = f(H_t)$ is some function of the history
  - The closer the agent’s model is to the true environment state, the better the agent will be able to strategize

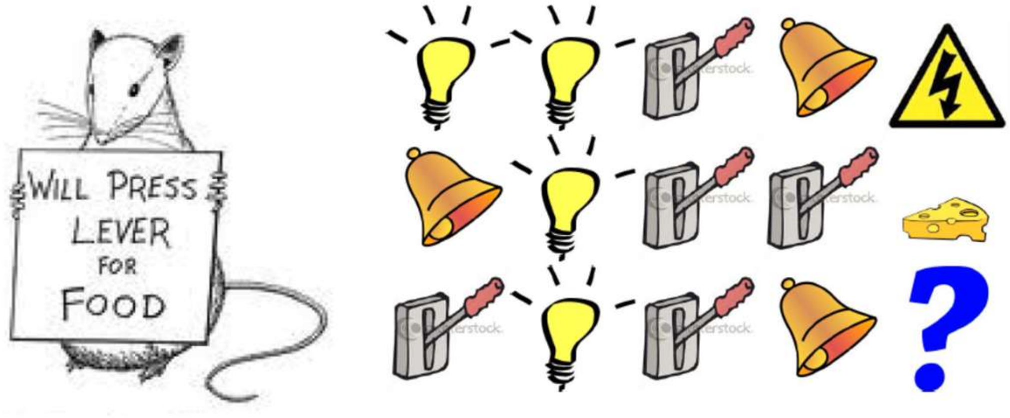
Defining Agent State

Image lifted from David Silver



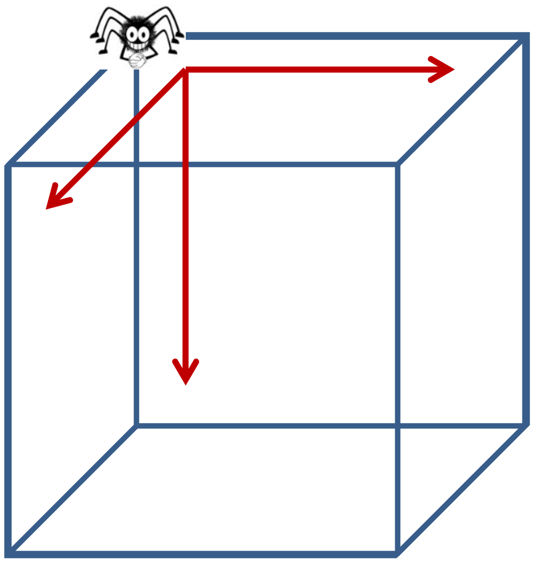
- What is the outcome?

Defining Agent State



- Different definitions of state result in different predictions
- *True* environment state not really known
  - Would greatly improve prediction if known

The World as we model It



Where the spider can go next
only depends on where she is

- Definition of Markov property:
  - The state of the system has a Markov property if the future only depends on the present

$$P(S_{t+1}|S_0, S_1, \dots, S_t) = P(S_{t+1}|S_t)$$

- States can be *defined* to have this property

A Markov *Process*

- A Markov *process* is a random process where the future is only determined by the present
  - Memoryless

- Is fully defined by the set of states $\mathcal{S}$, and the *state transition probabilities* $P(s_i \mid s_j)$
  - Formally, the tuple $\mathrm{M} = \langle \mathcal{S}, \mathcal{P} \rangle$.
  - $\mathcal{S}$ is the (possibly finite) set of states
  - $\mathcal{P}$ is the complete set of transition probabilities $P(s \mid s')$
  - Note $P(s \mid s')$ stands for $P(S_{t+1} = s | S_t = s')$ at any time t
  - Will use the shorthand $P_{s,s'}$

The transition probability

- For processes with a discrete, finite set of states, is generally arranged as *transition probability matrix*

$$\mathcal{P} = \begin{bmatrix} P_{s_1,s_1} & P_{s_2,s_1} & \cdots & P_{s_N,s_1} \\ P_{s_1,s_2} & P_{s_2,s_2} & \cdots & P_{s_N,s_2} \\ \vdots & \vdots & \ddots & \vdots \\ P_{s_1,s_N} & P_{s_2,s_N} & \cdots & P_{s_N,s_N} \end{bmatrix}$$

- More generally (for continuous-state processes, e.g. the state of an automobile), it is modelled as a parametric distribution

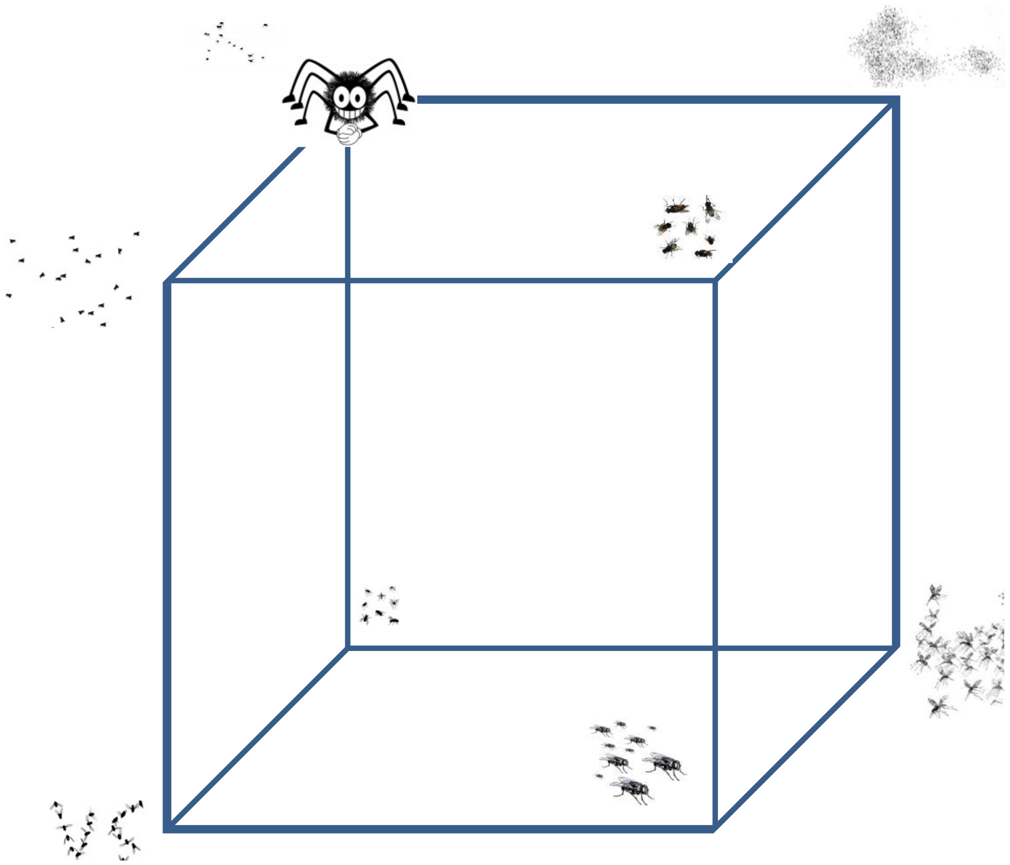
$$P_{s,s'} = f(s; \theta_{s'})$$

A Markov Reward Process

- A Markov *Reward* Process (MRP) is a Markov Process where states give you rewards

- At each state s, upon arriving at that state, you obtain a reward r, drawn from a distribution $P(r|s)$

Markov Reward Process



Reward: Upon arriving at any corner, the spider may catch a fly from the swarm hovering there

Rewards are corner specific and probabilistic: Different corners have different sized swarms with flies of different sizes. The spider only has a *probability* of catching a fly, but may not always catch one.

- Flider and the Markov reward process!

Markov Reward Process

- Formally, a Markov Reward Process is the tuple $\mathrm{M} = \langle \mathcal{S}, \mathcal{P}, \mathcal{R}, \gamma \rangle$
  - $\mathcal{S}$ is the (possibly finite) set of states
  - $\mathcal{P}$ is the complete set of transition probabilities $P_{s,s'}$
  - $\mathcal{R}$ is a *reward* function, consisting of the distributions $P(r|s)$
    - Or alternately, the expected value $R_s = E[r|s]$
  - $\gamma \in [0,1]$ is a *discount* factor

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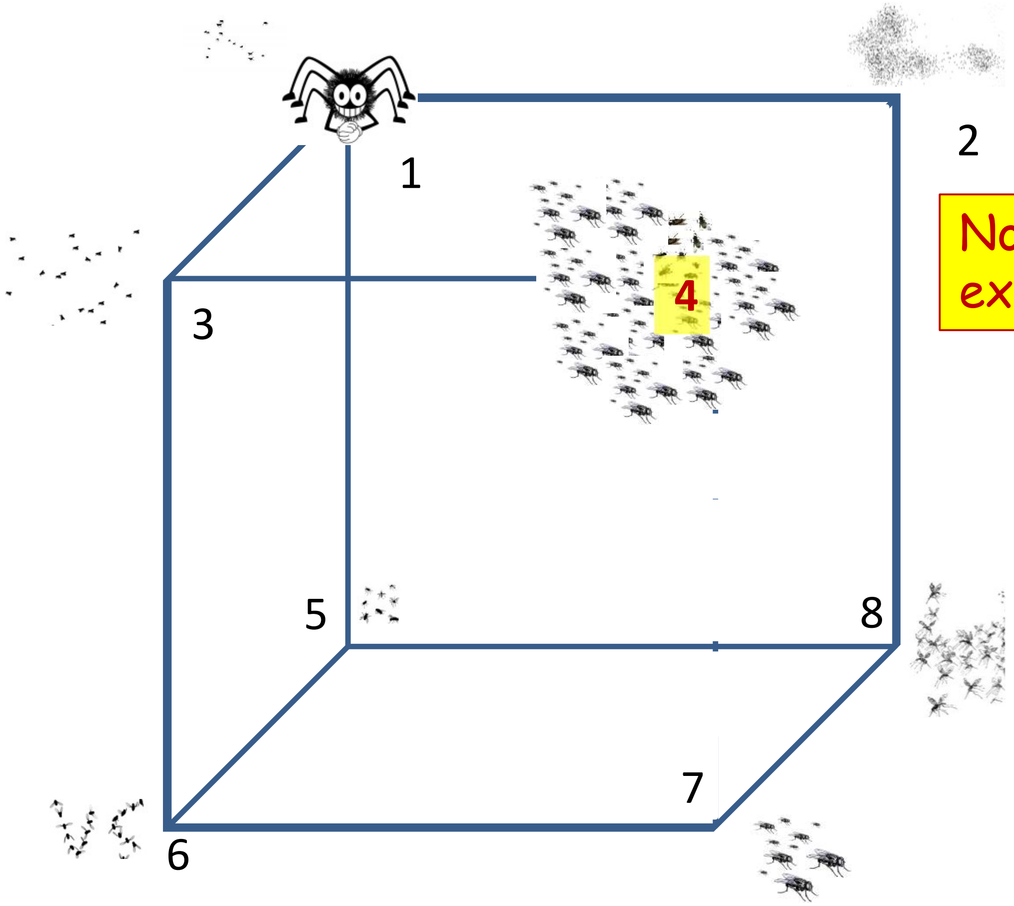
    - Or alternately, the **expected** value $R_s = E[r|s]$
  - $\gamma \in [0,1]$ is a *discount* factor *What on earth is this?*

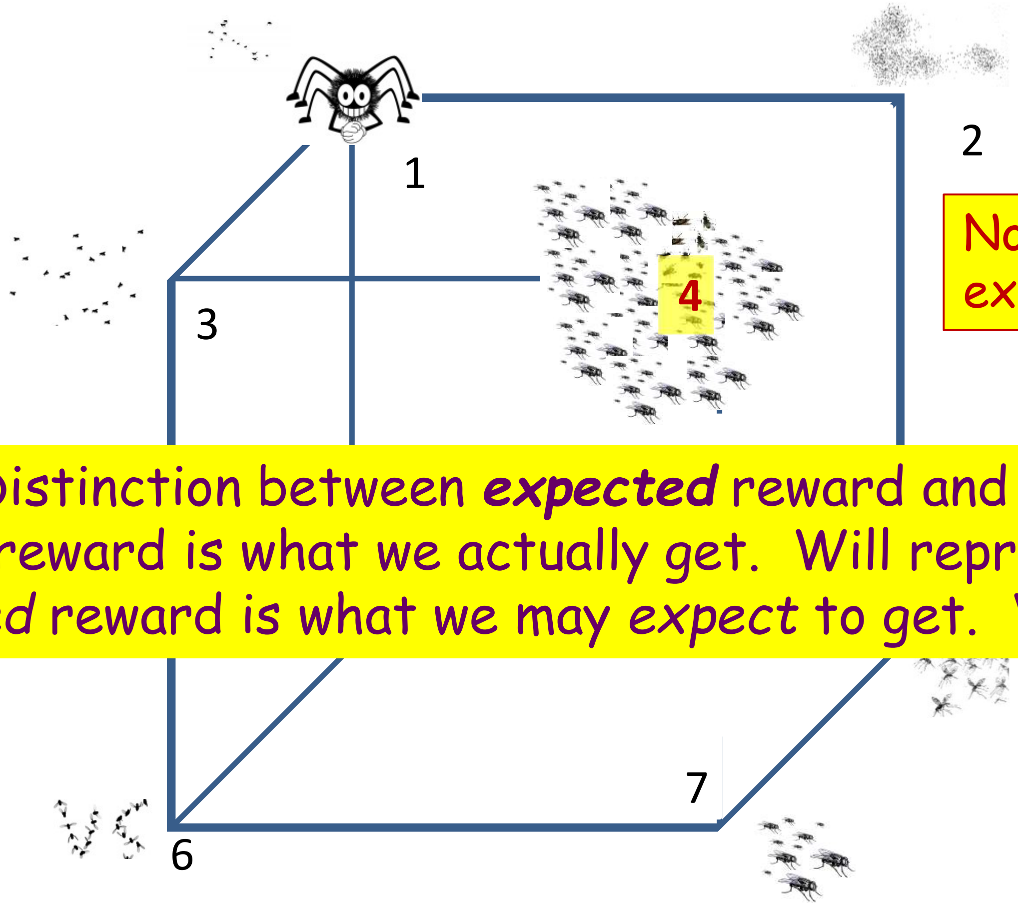
Rewards and *Expected* rewards



No route to corner 4 except from corner 3

- One step *expected* reward: R_1
  - Will this be greater if the spider heads to corner 2 or to corner 3?

Rewards and *Expected Rewards*

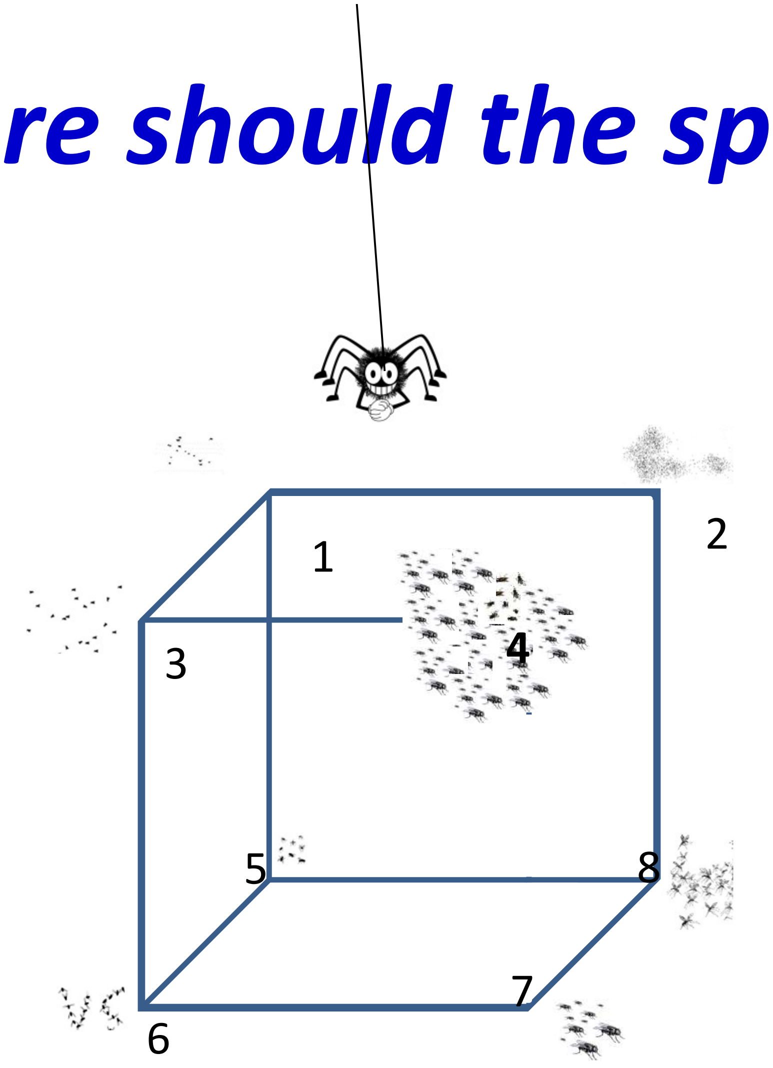


No route to corner 4 except from corner 3

Note: Distinction between ***expected*** reward and ***sample*** reward
Sample reward is what we actually get. Will represent by r
Expected reward is what we may *expect* to get. Will represent by R

- One step *expected* reward: R_1
  - Will this greater if the spider heads to corner 2 or to corner 3?

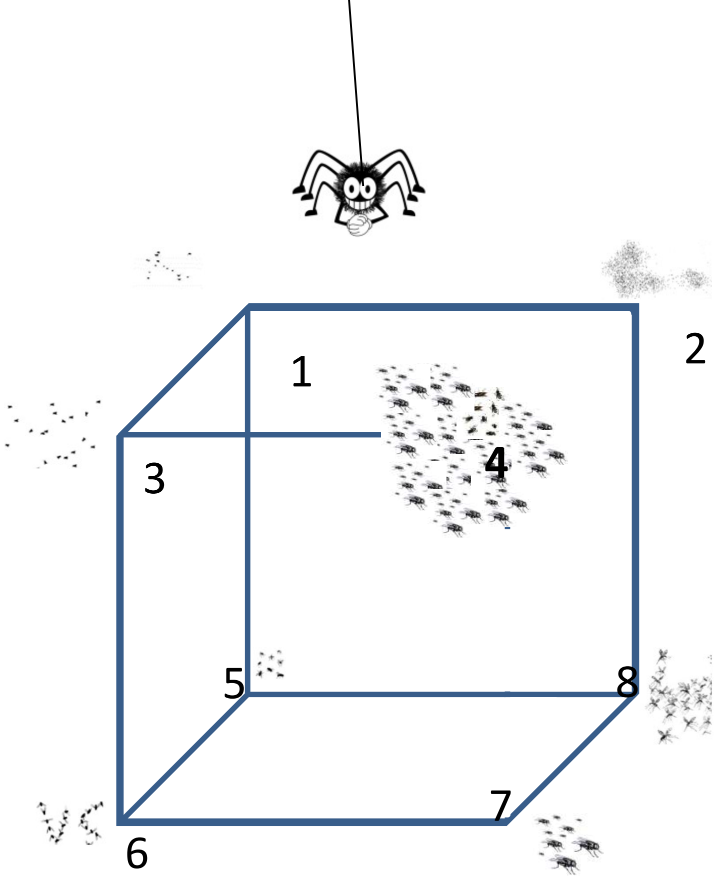
Where should the spider be?



- Flider has the option of landing on corner 1, 2 or 3 before she begins wandering the room
  - Which is the better corner to land on?

Where should the spider be?

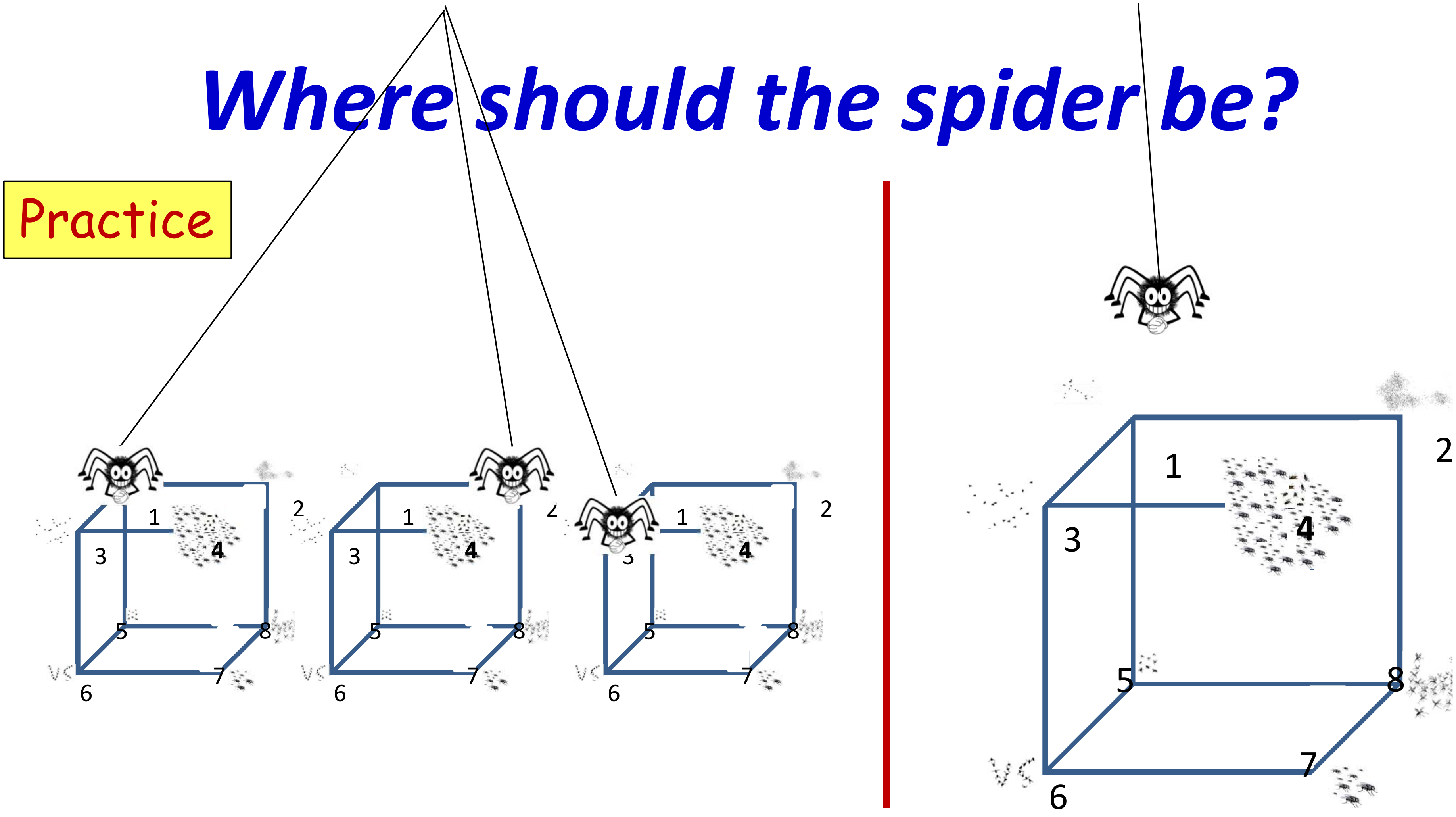
Need to know the *long-term consequences* of landing in the two corners



Where can she expect to get more food *in the long term?*

- Flider has the option of landing on corner 1, 2 or 3 before she begins wandering the room
  - Which is the better corner to land on?

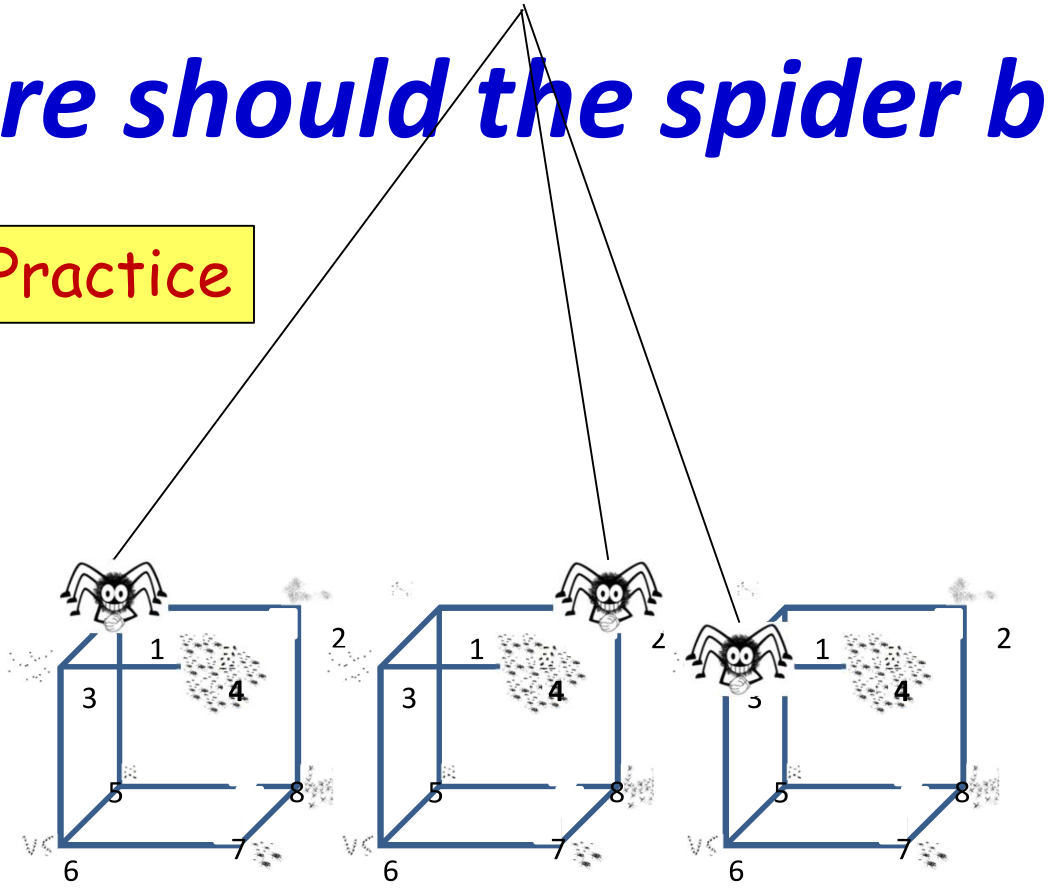
Where should the spider be?



- Assume she is allowed to "practice" once from each corner
  - To plan her future strategy

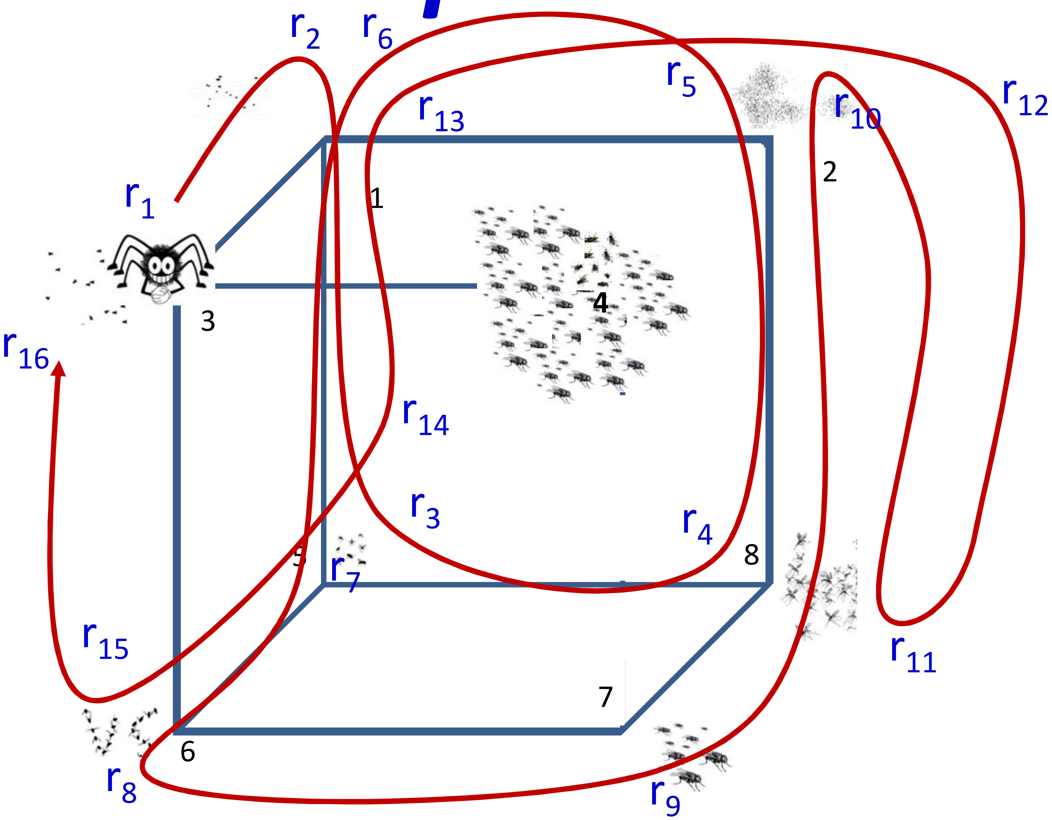
Where should the spider be?

Practice



- Must use her “practice” turn to assign a “value” to each of the corners
  - Guess how much food she would get in the long term from that corner

Flider practices



- Starting from 3, she gets r_1, r_2, r_3….
- Is $r_1 + r_2 + r_3$ … a realistic representation of what she'd get if she did it again?

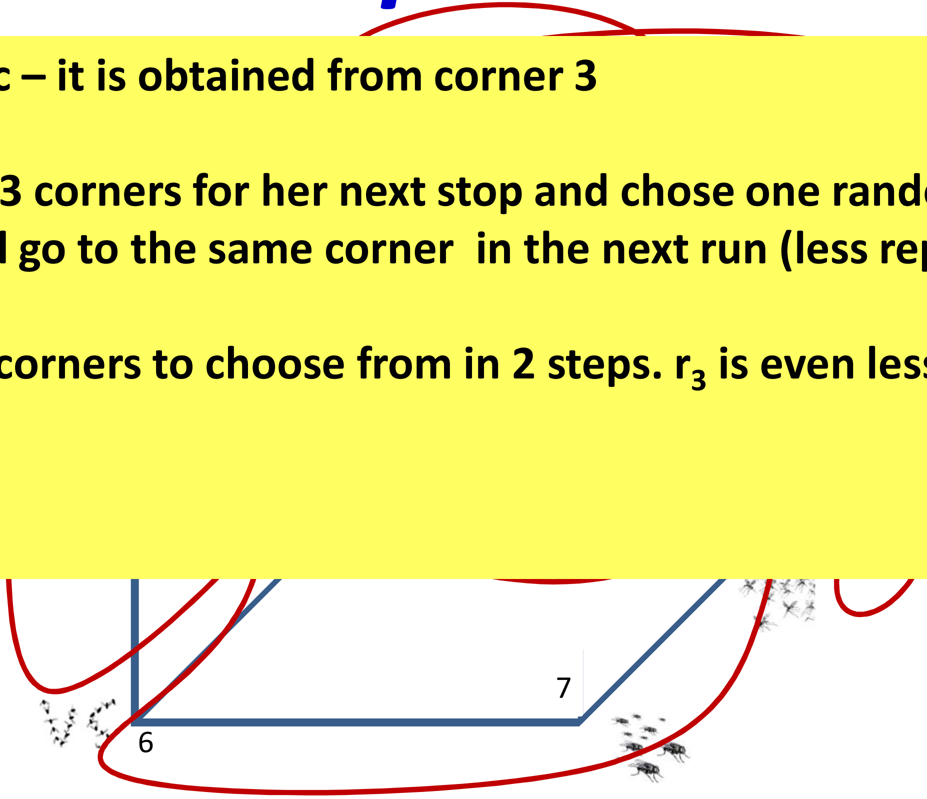
Flider practices

r_1 is somewhat realistic – it is obtained from corner 3

r_2: she had a choice of 3 corners for her next stop and chose one randomly during practice. Unlikely she'll go to the same corner in the next run (less representative)

r_3: she had 9 possible corners to choose from in 2 steps. r_3 is even less representative of future runs

And so on…



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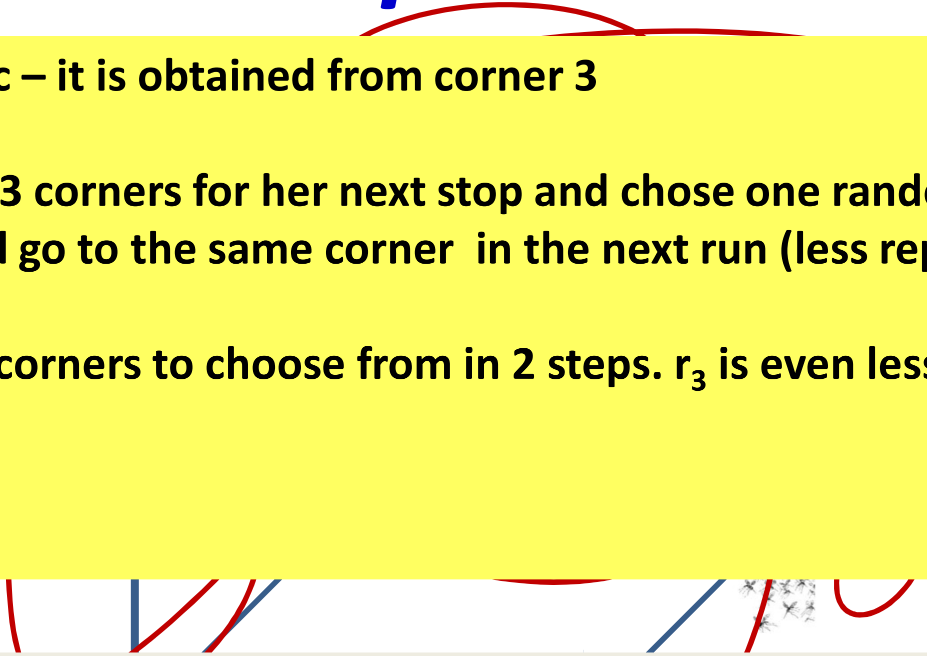

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And so on…

A better guess for how good it is to land at "3":

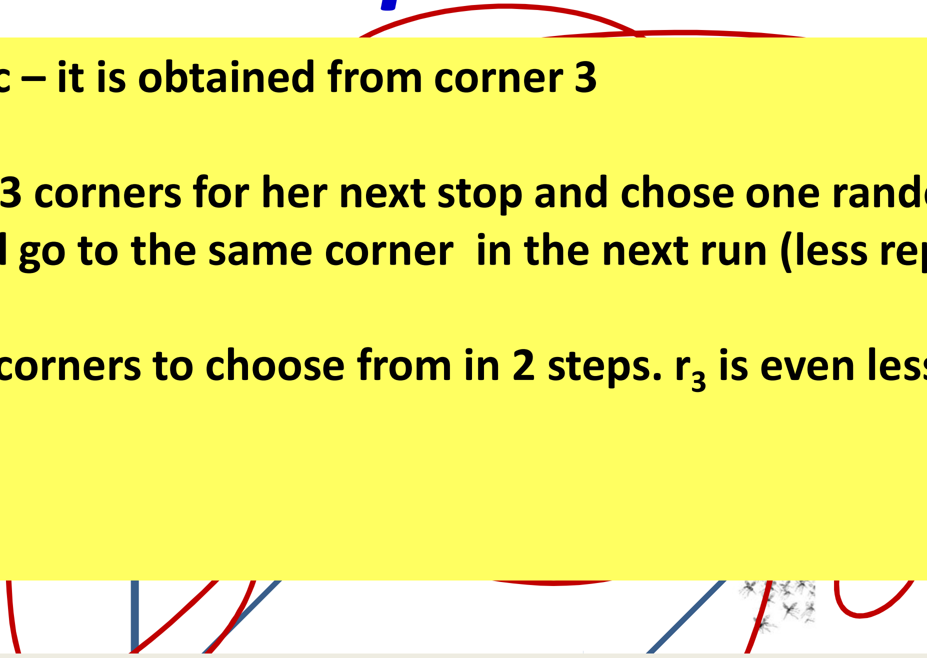
$$r_1 + a_1 r_2 + a_2 r_3 + a_3 r_4 + \cdots$$

Where $0 \le a_i \le 1$

(you "trust" the readings from farther in the future less)

- Is $r_1 + r_2 + r_3$ … a realistic representation of what she'd get if she did it again?

Flider practices



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Where $\boldsymbol{0 \le a_i \le 1}$
(you "trust" the readings from farther in the future less)

- A "mathematically good" choice: $a_i = \gamma^i$ where $0 \le \gamma \le 1$ hat

she'd get if she did it again?

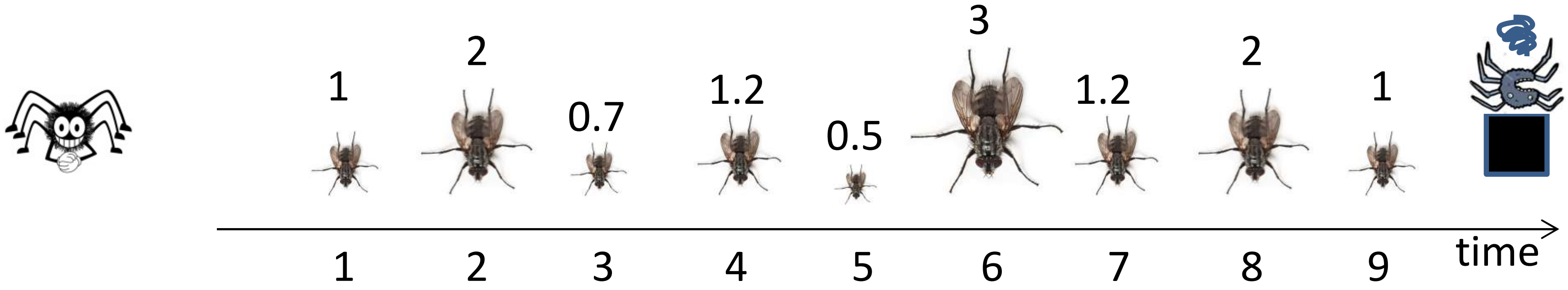
The discounted return

$$G_t = r_{t+1} + \gamma r_{t+2} + \gamma^2 r_{t+3} + \cdots = \sum_{k=0}^{\infty} \gamma^k r_{t+k+1}$$

- The *return* is the total *future* reward all the way to the end
- But each future step is slightly less “believable” and is hence discounted
  - We trust our own observations of the future less and less
    - The future is a fuzzy place

- The discount factor γ is our belief in the predictability of the future
  - $\gamma = 0$: The future is totally unpredictable, only trust what you see immediately ahead of you (myopic)
  - $\gamma = 1$: The future is clear; consider all of it (far sighted)
- **Part of the Markov Reward Process model**

Rewards

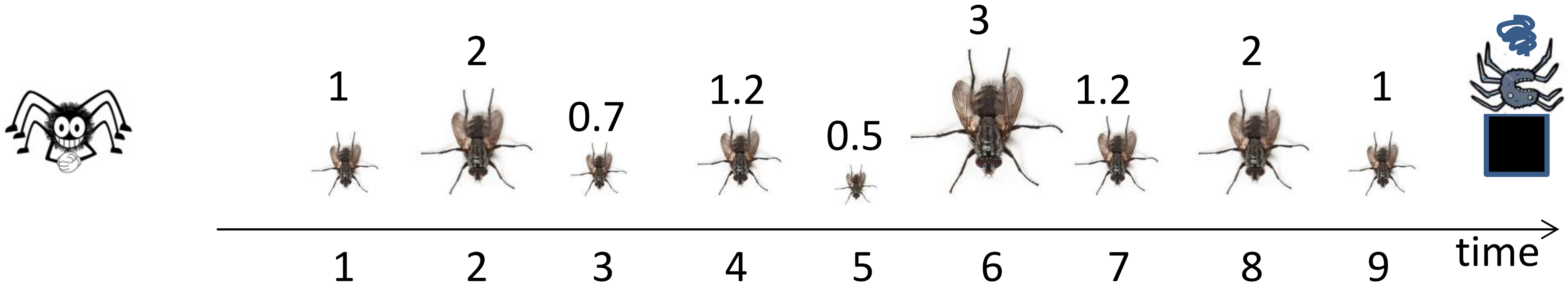


- Our spider goes wandering..

  $r_1 = 1, r_2 = 2, r_3 = 0.7, r_4 = 1.2, r_5 = 0.5, \ldots$

- Are these *sample rewards* or *expected rewards?*

Returns

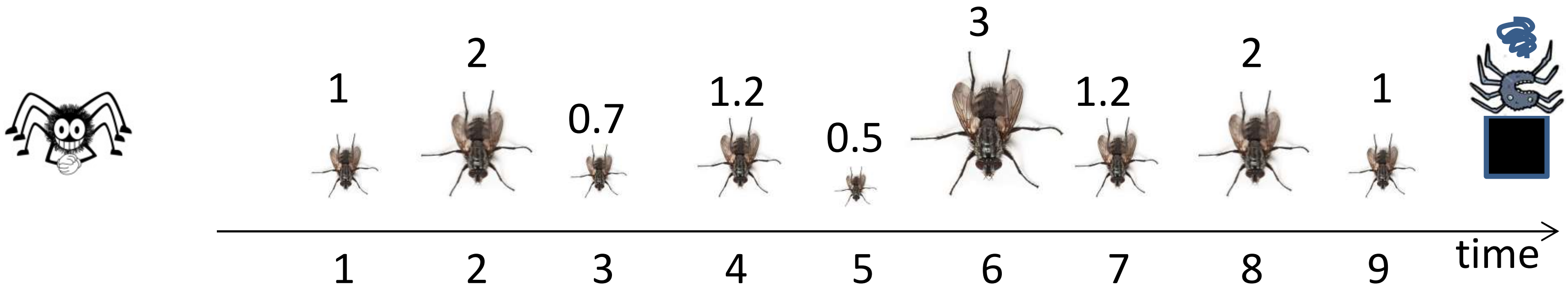


- Our spider goes wandering..

$$r_1 = 1, r_2 = 2, r_3 = 0.7, r_4 = 1.2, r_5 = 0.5, \ldots$$

- We decide the discounting factor $\gamma = 1$
  - Really trusting the future
- What is the return G_t at $t = 1$?

Returns



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- We decide the discounting factor $\gamma = 1$
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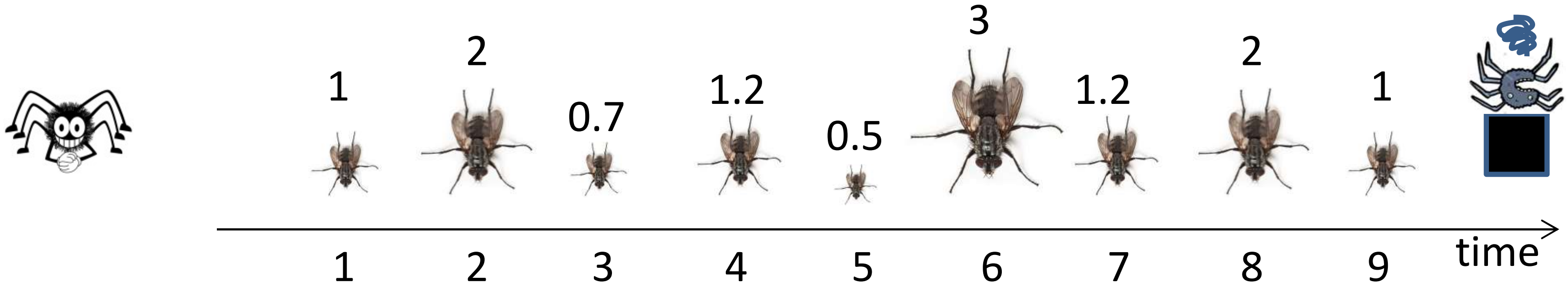
- What is the return G_t at $t = 1$?
- What is the return G_t at $t = 7$?

Returns



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- We decide the discounting factor $\gamma = 1$
  - Really trusting the future
- What is the return G_t at $t = 1$?
- What is the return G_t at $t = 7$?
- Are these *sample* returns or *expected* returns?

Returns

- Discounted sample returns G_t by themselves carry a fuzzy meaning
  - Why should we discount something we already observed?
- However, they make sense as *samples* of the possible future when you are at any state
  - If you are at any state, what is the *expected* return $E[G_t]$

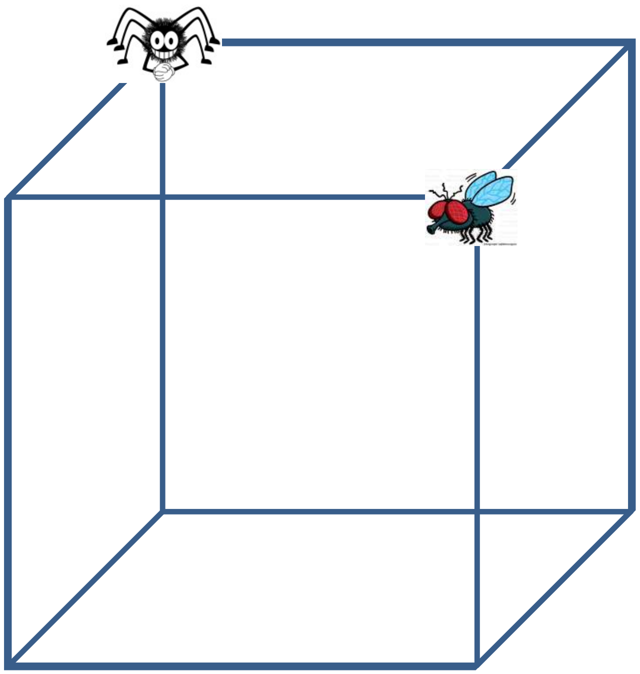
Introducing the “Value” function

- The “Value” of a state is the expected total discounted return, starting from that state

$$V_s = E[G|S = s]$$

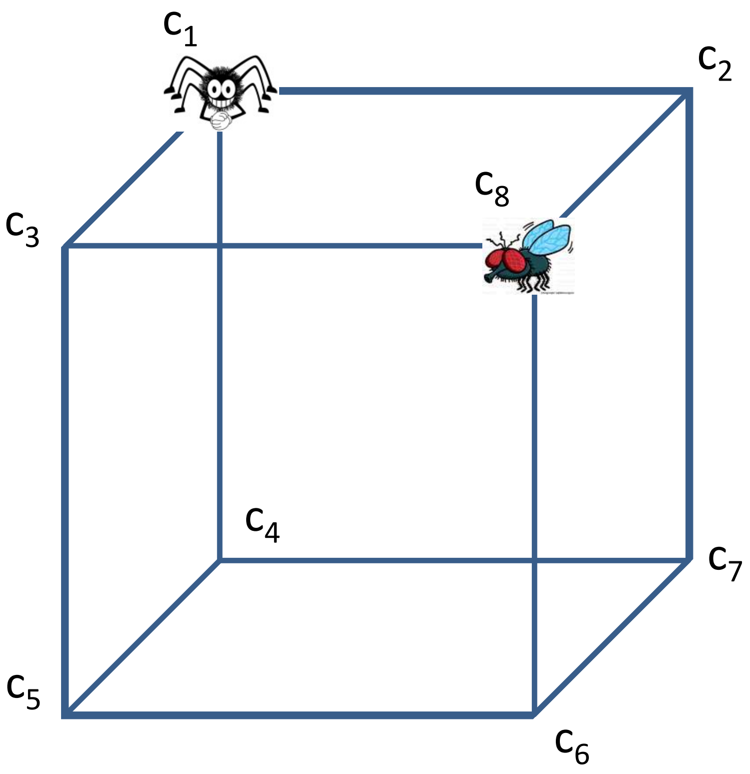
- This is not a function of time
  - i.e. it doesn’t matter *when* you arrive at s, the expected return from that point on is V_s

The spider again



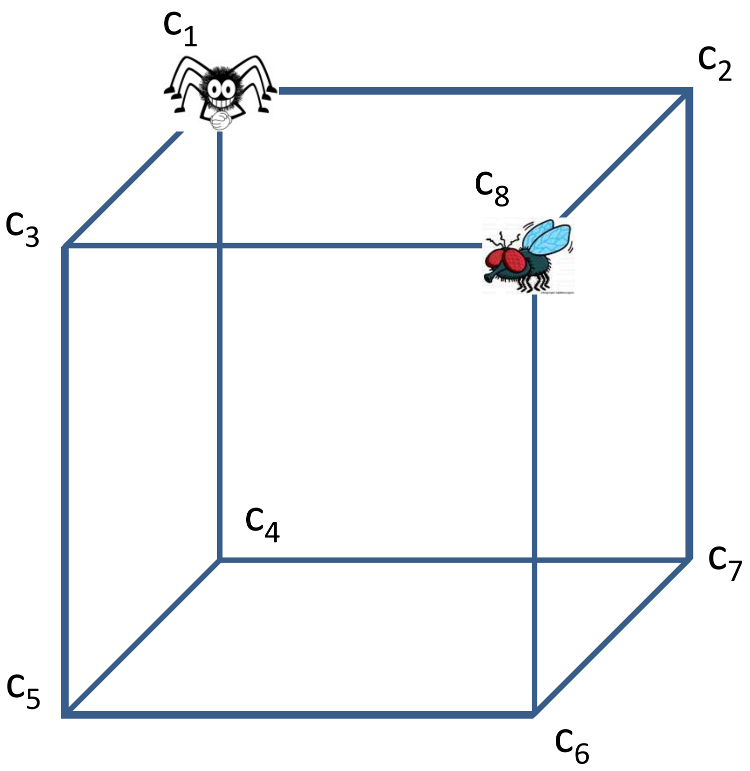
- The spider gains a reward of value 1 if she consumes the fly
- The spider has infinite patience
- What is the value of starting at each corner?

The spider again



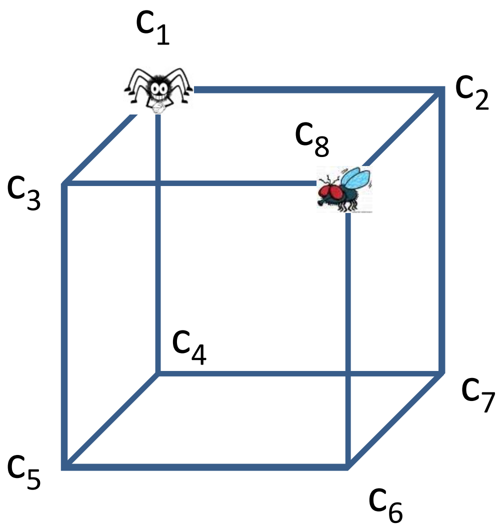
- Regardless of which corner the spider starts at, she will eventually, randomly, nab the fly
- The expected return from any corner is 1!
- The *value of being at any corner is 1 for all corners*

The *hungry* spider



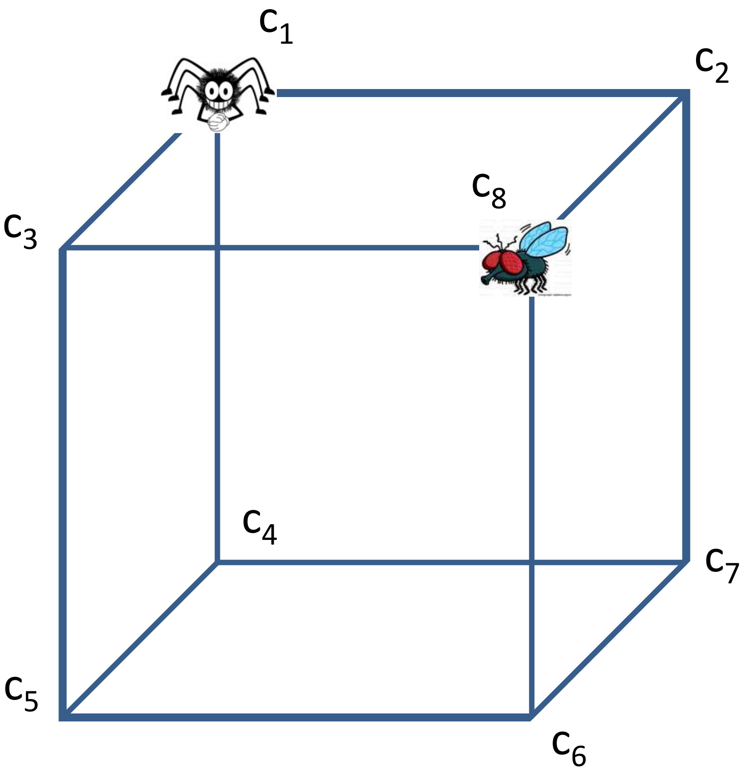
- The spider is hungry
- She gets a negative reward of -1 for every minute spent finding food
- What is the expected return if she starts at c_1

The *hungry* spider



- Posing the problem: There is a total reward/penalty associated with each corner
  - -1 if the corner has no fly
    - Will definitely spend at least one more minute hunting
  - 1 at the corner that has the fly (satisfied!)
- Thus $r_{c_x} = -1$ for $c_1 \dots c_7$
- $r_{c_8} = 1$
- Note: We could also assign costs/rewards to edges in addition to nodes, if we want more detail, but won't do so for our lectures

The *hungry* spider



$$V_{c_1} = -1 + \frac{1}{3}V_{c_2} + \frac{1}{3}V_{c_3} + \frac{1}{3}V_{c_4}$$

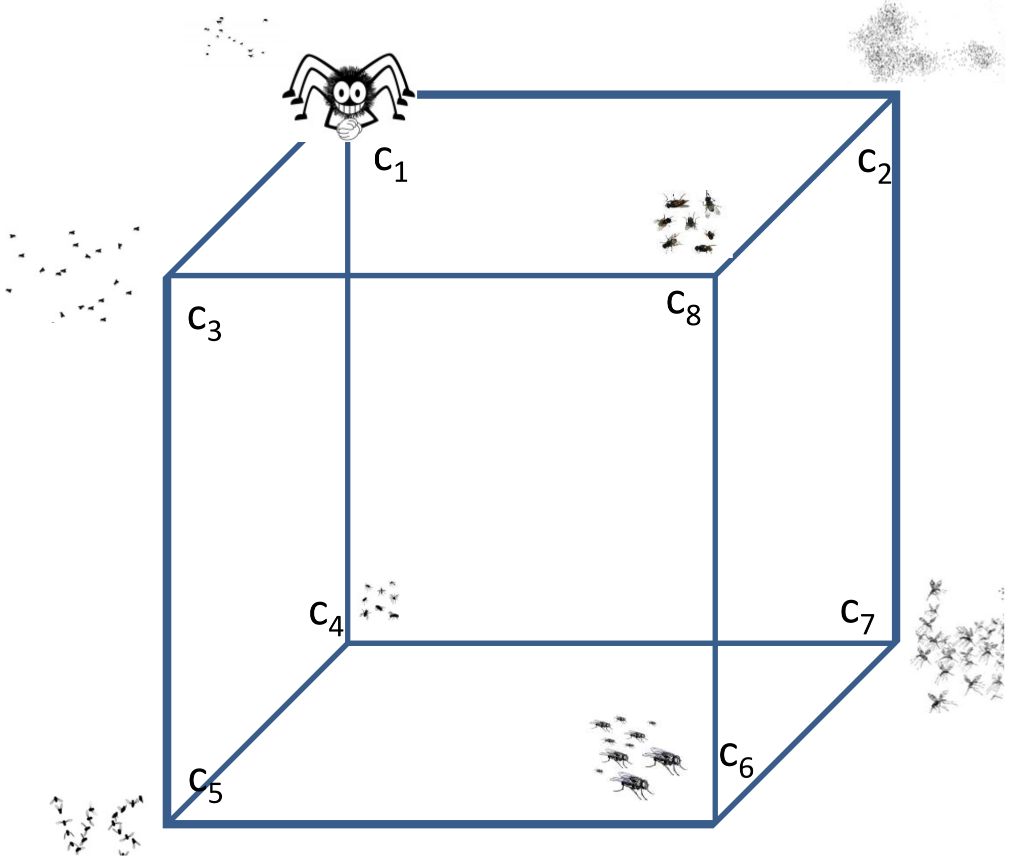
$$V_{c_2} = -1 + \frac{1}{3}V_{c_1} + \frac{1}{3}V_{c_7} + \frac{1}{3}V_{c_8}$$

$$\vdots$$

$$V_{c_8} = 1$$

- A familiar solution
- Assuming $\gamma = 1$
  - A natural fit in this problem

More generally



$$V_{c_1} = R_{c_1} + \gamma\left(\frac{1}{3}V_{c_2} + \frac{1}{3}V_{c_3} + \frac{1}{3}V_{c_4}\right)$$

$$V_{c_2} = R_{c_2} + \gamma\left(\frac{1}{3}V_{c_1} + \frac{1}{3}V_{c_7} + \frac{1}{3}V_{c_8}\right)$$

$$\vdots$$

$$V_{c_8} = R_{c_8} + \gamma\left(\frac{1}{3}V_{c_2} + \frac{1}{3}V_{c_3} + \frac{1}{3}V_{c_6}\right)$$

- A familiar solution

The Bellman Expectation Equation

- The value function of a state is the *expected discounted return*, when the process begins at that state

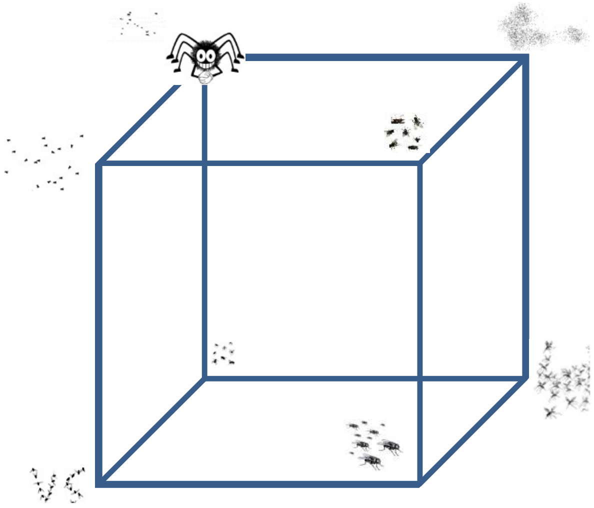
$$G_t = \sum_{k=0}^{\infty} \gamma^k r_{t+k+1}$$

$$V_s = E[G|S = s]$$

- **The Bellman Expectation Equation:**

$$\boxed{V_s = R_s + \gamma \sum_{s'} P_{s',s} V_{s'}}$$

Why discounted return?



- In processes with infinite horizon, which can go on for ever, the total undiscounted return will be infinite for every path $\sum_{k=0}^{\infty} r_{t+k+1}$ will be infinite for every path
  - For finite horizon processes, a discount factor $\gamma = 1$ is good. It lets us talk in terms of actual total return
  - For infinite horizon processes, discounting $\gamma < 1$ is required for meaningful mathematical analysis : $\sum_{k=0}^{\infty} \gamma^k r_{t+k+1}$

The Bellman Expectation Equation

$$V_s = R_s + \gamma \sum_{s'} P_{s',s} V_{s'}$$

$$\begin{bmatrix} V_{s_1} \\ V_{s_2} \\ \vdots \\ V_{s_N} \end{bmatrix} = \begin{bmatrix} R_{s_1} \\ R_{s_2} \\ \vdots \\ R_{s_N} \end{bmatrix} + \gamma \begin{bmatrix} P_{s_1,s_1} & P_{s_2,s_1} & \cdots & P_{s_N,s_1} \\ P_{s_1,s_2} & P_{s_2,s_2} & \cdots & P_{s_N,s_2} \\ \vdots & \vdots & \ddots & \vdots \\ P_{s_1,s_N} & P_{s_2,s_N} & \cdots & P_{s_N,s_N} \end{bmatrix} \begin{bmatrix} V_{s_1} \\ V_{s_2} \\ \vdots \\ V_{s_N} \end{bmatrix}$$

$$\mathcal{V} = \mathcal{R} + \gamma \mathcal{P} \mathcal{V}$$

- Bellman expectation equation in matrix form

The Bellman Expectation Equation

$$\mathcal{V} = \mathcal{R} + \gamma \mathcal{P} \mathcal{V}$$

- Given the MRP $\mathrm{M} = \langle \mathcal{S}, \mathcal{P}, \mathcal{R}, \gamma \rangle$
    - I.e. the expected rewards at every state, and the transition probability matrix,
  - the value functions for all states can be easily computed through matrix inversion

$$\mathcal{V} = (I - \gamma \mathcal{P})^{-1} \mathcal{R}$$

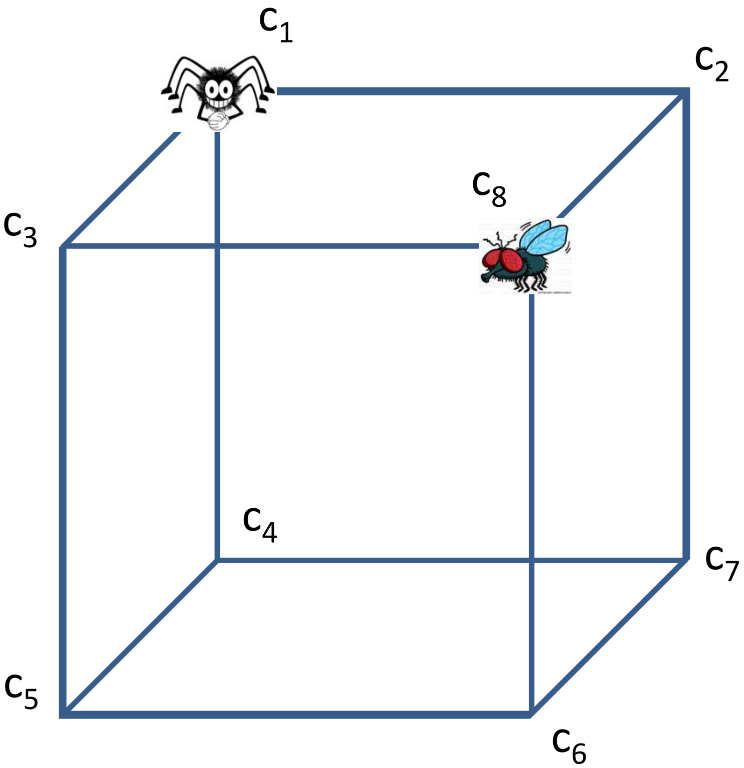
- Finding the values of states is a key problem in planning and reinforcement learning

- Unfortunately, for very large state spaces, the above matrix inversion is not tractable
  - Also not invertible for small state spaces if $\gamma = 1$
    - Inversion cannot be used to find $\mathcal{V}$ even when it is finite (e.g. our fly problem), if $\gamma = 1$
- Much of what we will deal with is how to tackle this problem

Moving on..

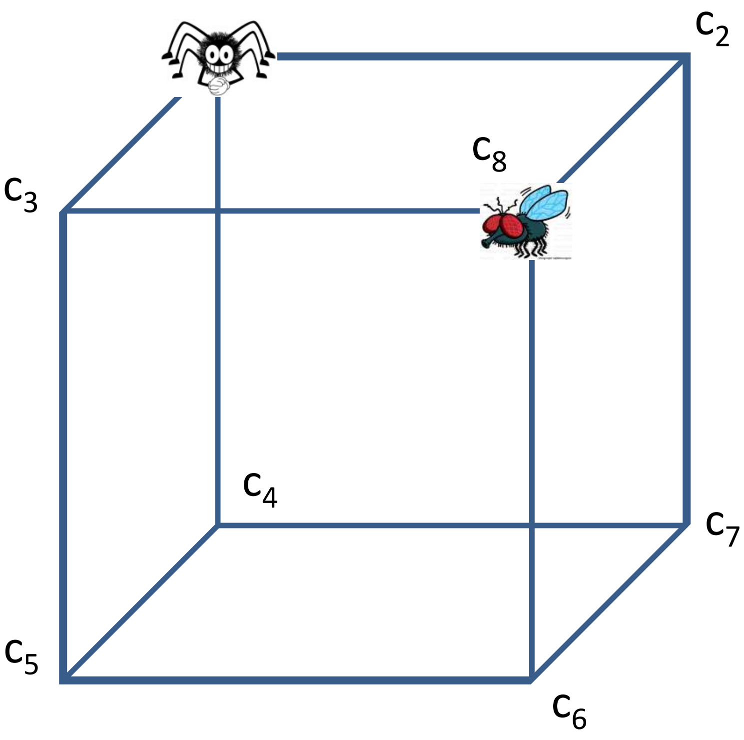
- Up next … Markov *Decision* Processes

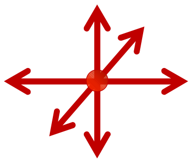
MDP



- We have assumed so far that the agent behaves randomly
  - The agent has no *agency*
  - Let’s make the agent more intelligent..

A more realistic problem



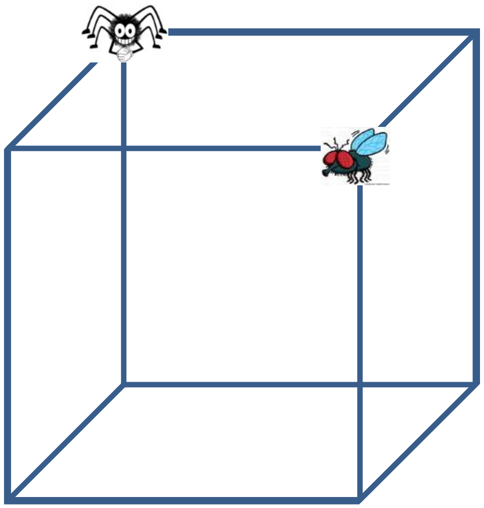


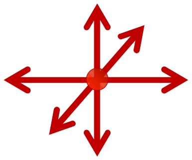
Full set of possible actions

How do we model this system?

- The spider actively chooses which way to move
  - The agent *takes action*
  - Ideally, it would move in the general direction of the fly
- However, each time the spider moves, the fly jumps up and settles at another corner
  - The agent's action changes the environment!

Redefining the problem

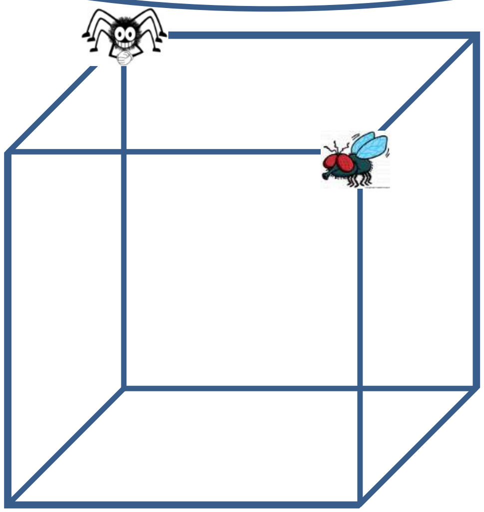


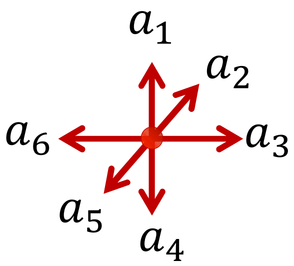


Full set of possible actions

- Each time the spider moves in any direction, the fly randomly jumps
- The fly arrives at a new state but ..
  - The state it arrives in depends on where the fly jumped
  - Which depends on which direction the Spider moved
- The spider's action *modifies the state transition probabilities!!*

What is $P^{a}_{s,s'}$





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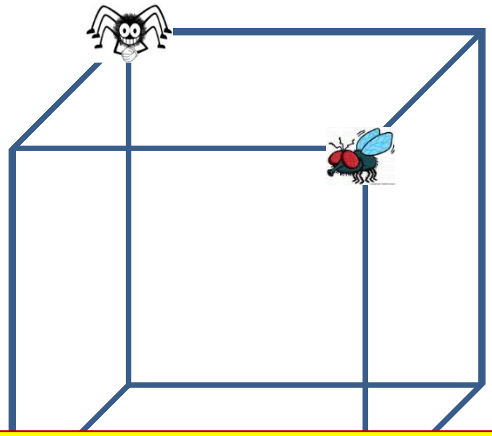
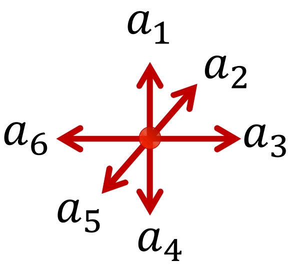
What is $P^{a}_{s,s'}$





Full set of possible actions

Must modify our notion of states and actions,
and define the behavior of the fly, to characterize.

- Each time the spider moves in any direction, the fly randomly jumps
- The fly arrives at a new state but ..
  - The state it arrives in depends on where the fly jumped
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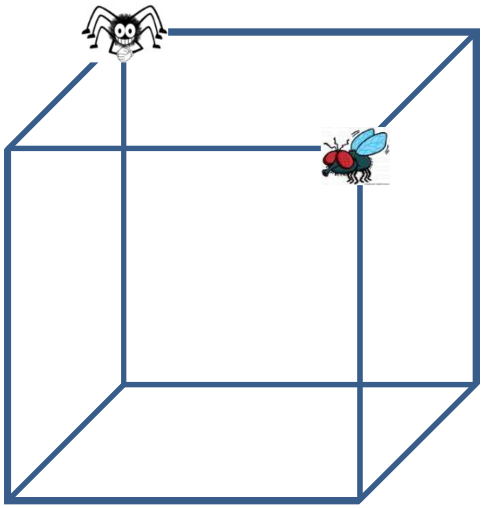
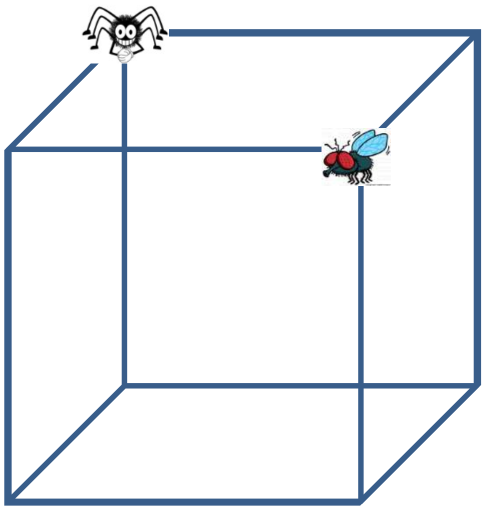
- The spider’s action *modifies the state transition probabilities!!*

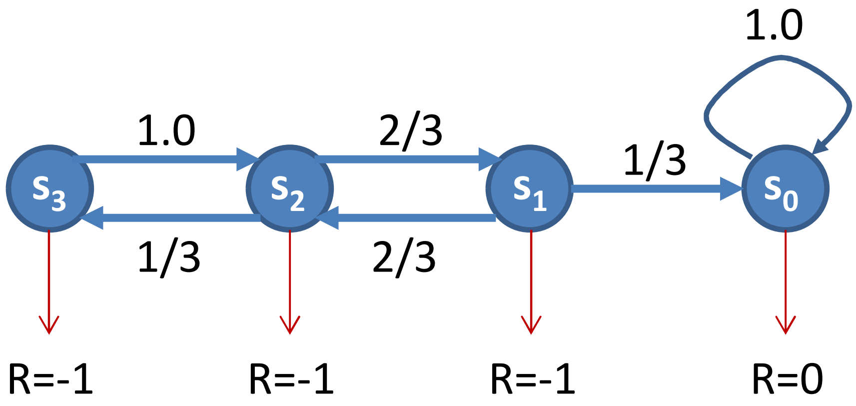
Trick Question: Redefining the States



- There are, in fact, only four states, not eight
  - Manhattan distance between fly and spider = 0 (s_0)
  - Distance between fly and spider = 1 (s_1)
  - Distance between fly and spider = 2 (s_2)
  - Distance between fly and spider = 3 (s_3)
- Can, in fact, redefine the MRP entirely in terms of these 4 states
- There are two actions a+ and a-
- Need an idea of the behavior of the fly

The Fly Markov Reward Process





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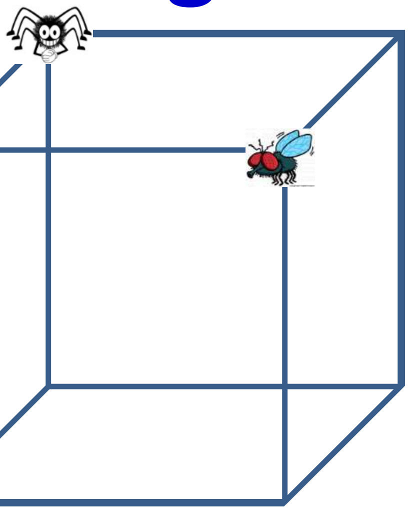
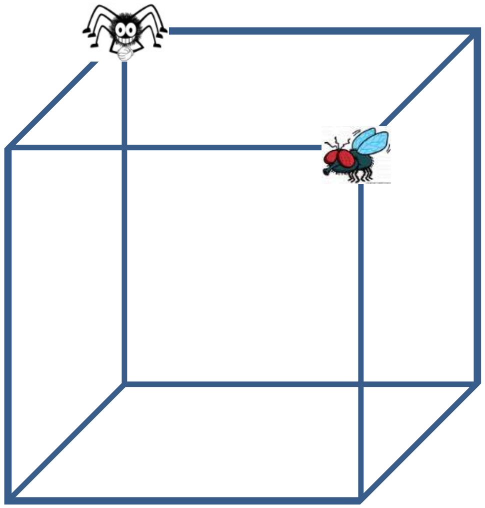
The Markov *Decision* Process: Defining Actions



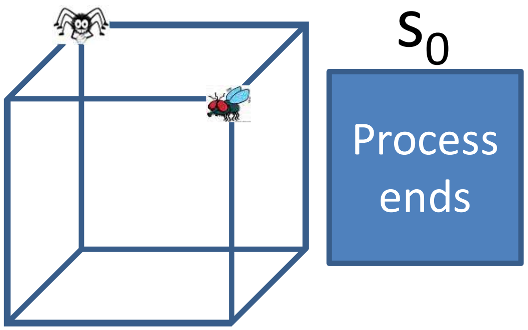
- Two types of actions:
  - a_+ : Increases distance to fly by 1
  - a_- : Decreases distance to fly by 1

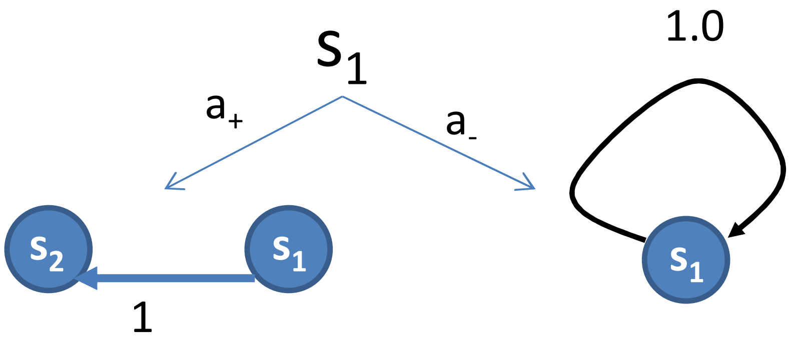
The Fly Markov Decision Process

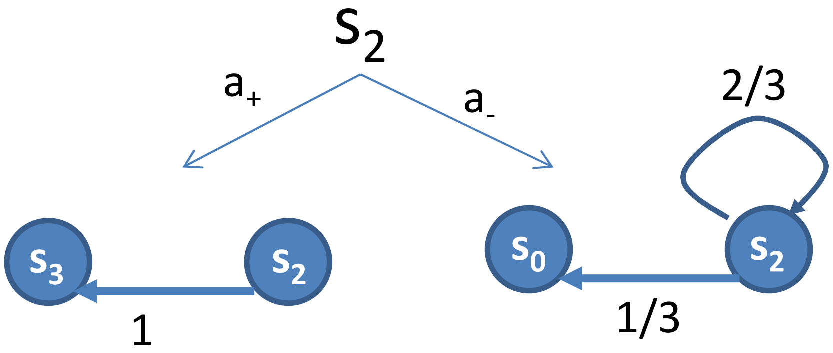


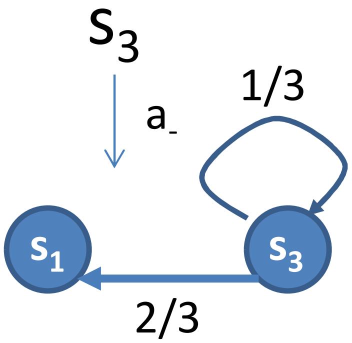
- The behavior of the fly:
  - If the spider is moving *away from it,* it does nothing
  - If the spider is moving *towards* it, it randomly hops to a different adjacent corner
    - 2/3 of the time, it increases the distance to the fly by 1
    - 1/3 of the time, it *decreases* the distance to the fly by 1

The Fly Markov Decision Process

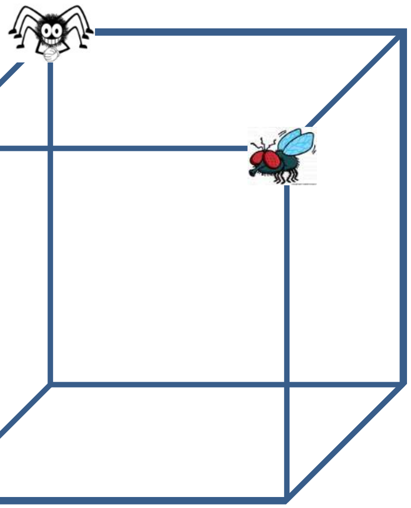


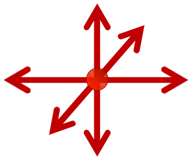






Redefining the problem





Full set of possible actions

- Each time the spider moves in any direction, the fly randomly jumps

Note: This is a simile for many problems in life, e.g. driving, stock market, advertising, etc.
The agents actions modifies how the environment behaves

  - Which depends on which direction the spider moved
- The spider’s action *modifies the state transition probabilities!!*

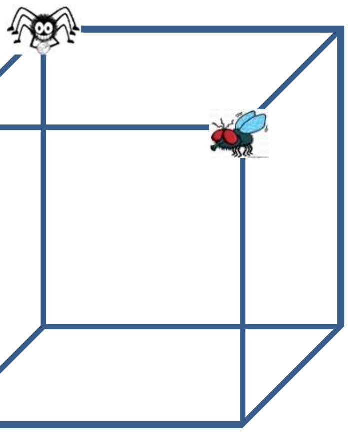
The Markov Decision Process

- A ***Markov Decision Process*** is a Markov Reward Process, where the agent has the ability to decide its actions!
  - We will represent individual actions as a
  - We will represent the action at time t as A_t
- The agent's actions affect the environment's behavior
  - The transitions made by the environment are functions of the action
  - The rewards returned are functions of the action

The Markov Decision Process

- Formally, a Markov Decision Process is the tuple $\mathrm{M} = \langle \mathcal{S}, \mathcal{P}, \mathcal{A}, \mathcal{R}, \gamma \rangle$
  - $\mathcal{S}$ is a (possibly finite) set of states : $\mathcal{S} = \{s\}$
  - $\mathcal{A}$ is a (possibly finite) set of *actions* : $\mathcal{A} = \{a\}$
  - $\mathcal{P}$ is the set of *action conditioned* transition probabilities $P_{s,s'}^{a} = P(S_{t+1} = s | S_t = s', A_t = a)$
  - $\mathcal{R}$ is an *action conditioned reward* function
$$R_s^a = E[r | S = s, A = a]$$
  - $\gamma \in [0,1]$ is a *discount* factor

Introducing: Policy



- The *policy* is the probability distribution over actions that the agent may take at any state

$$\pi(a|s) = P(A_t = a|S_t = s)$$

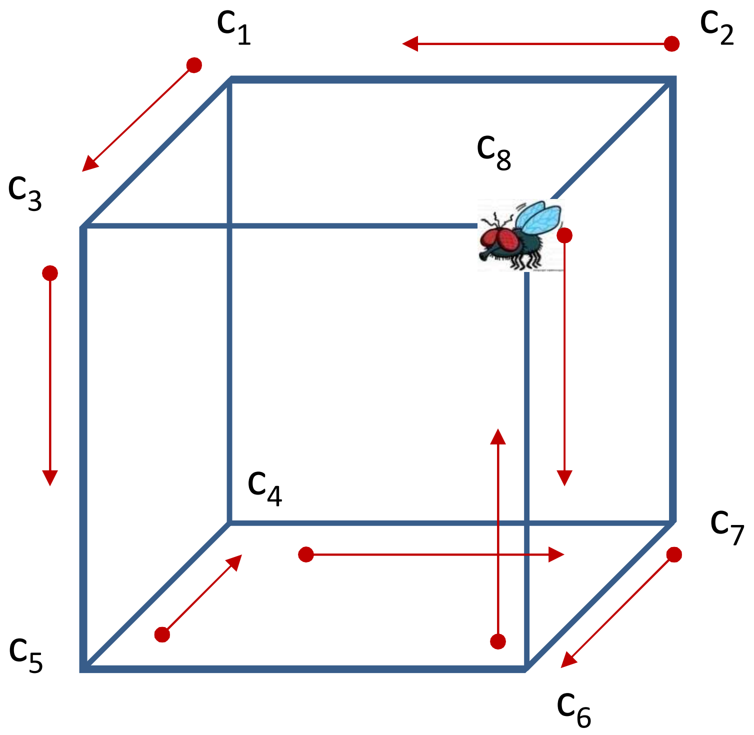
  - What are the preferred actions of the spider at any state

- The policy may be deterministic, i.e.

$$\pi(a|s) = 1\ for\ a = a_s; \quad 0\ for\ a \neq a_s$$

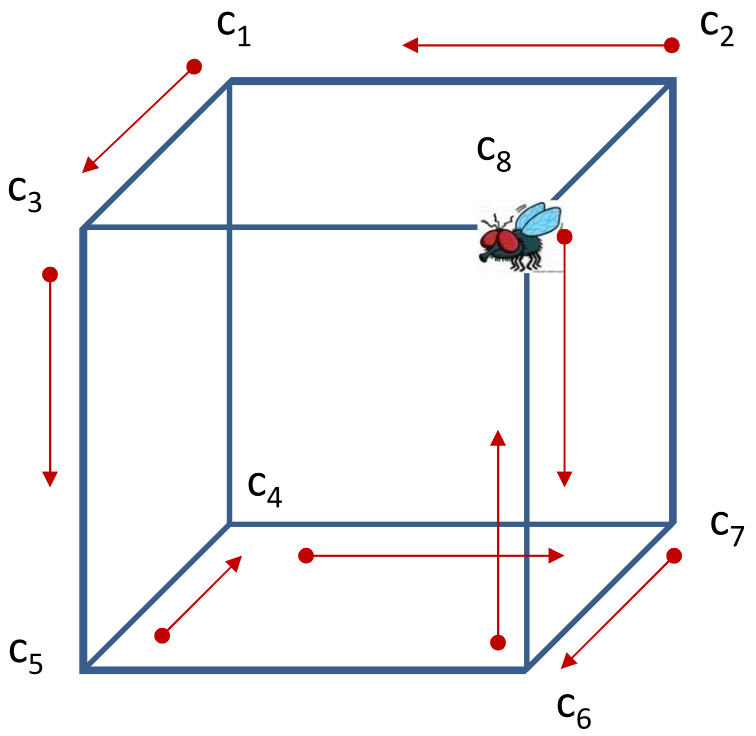
  - where a_s is the preferred action in state s

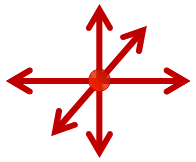
An example of a policy



- Assuming the fly does not move
  - This example is not a particularly good policy for the spider

An example of a policy

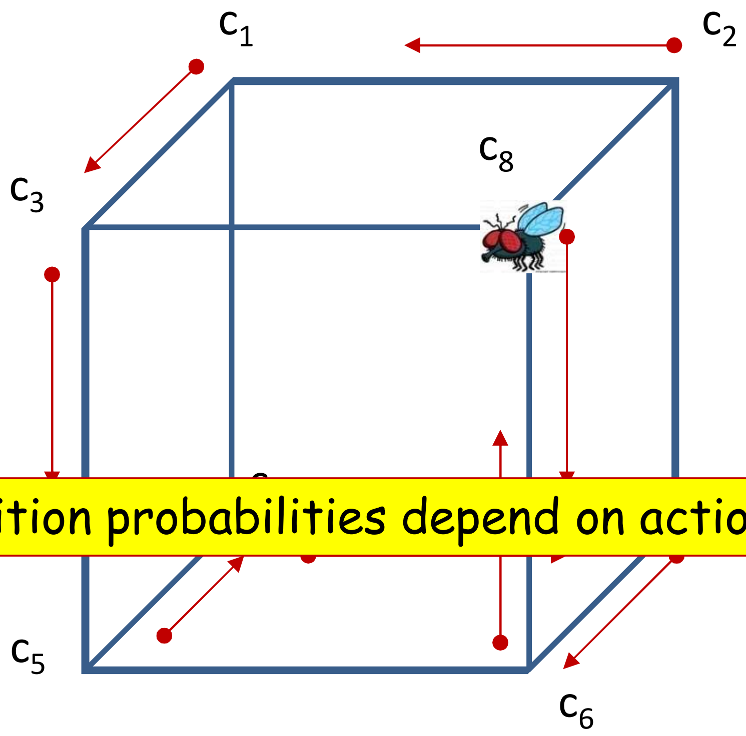




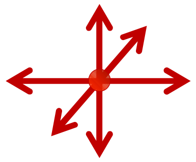
Full set of possible actions

- What are the (action dependent) transition probabilities of the states here?

An example of a policy



The transition probabilities depend on actions, but not on policy



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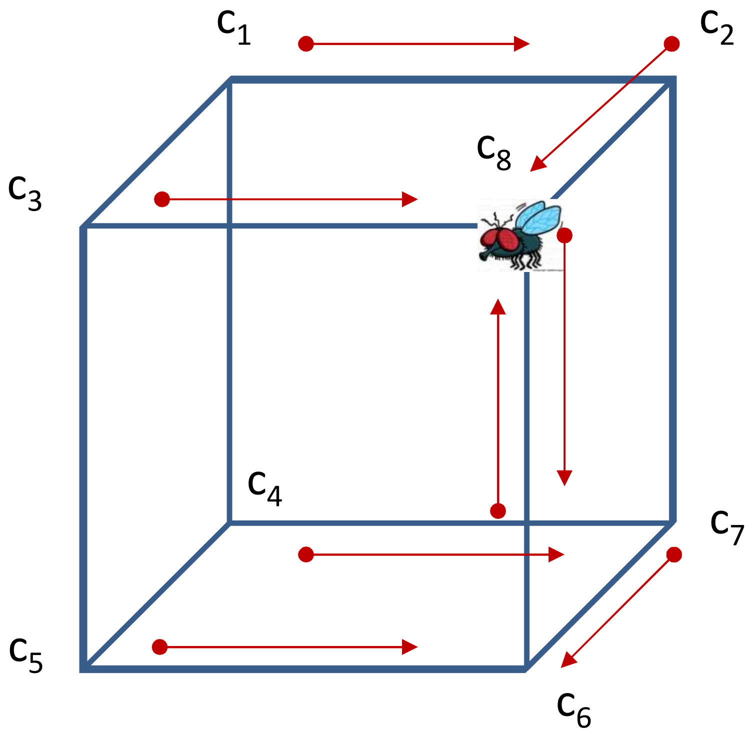
An example of a policy

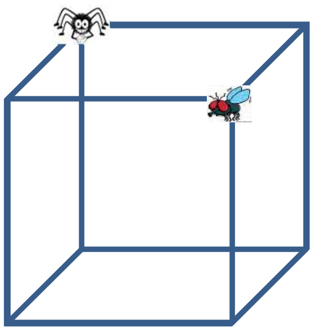


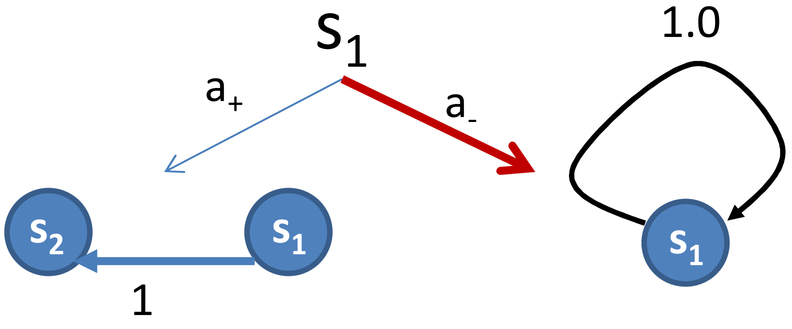
- Assuming the fly does not move
  - This is a different *optimal* policy
  - What are the transition probabilities here?

The value function of an MDP

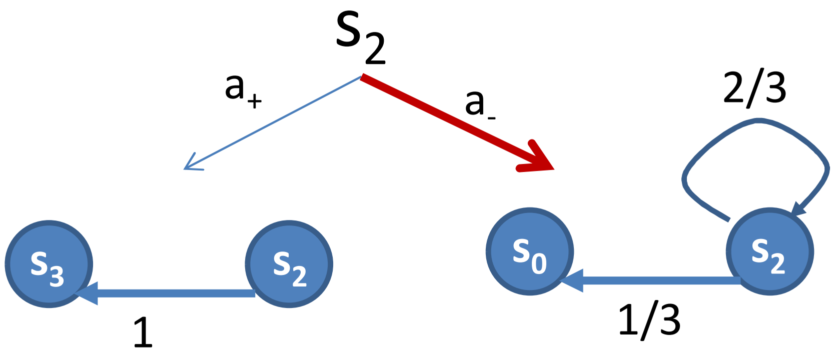
- The *expected return* from any state depends on the policy you follow

The Fly MDP: Policy 1

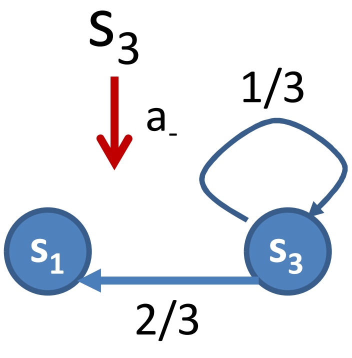




$$V_{s_1} = R_{s_1} + \gamma V_{s_1}$$

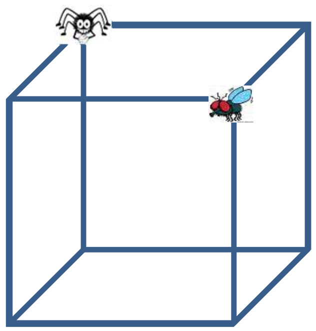


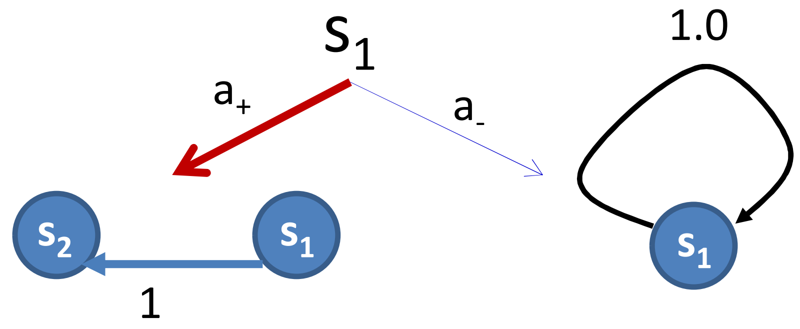
$$V_{s_2} = R_{s_2} + \gamma\left(\frac{1}{3}V_{s_0} + \frac{2}{3}V_{s_2}\right)$$



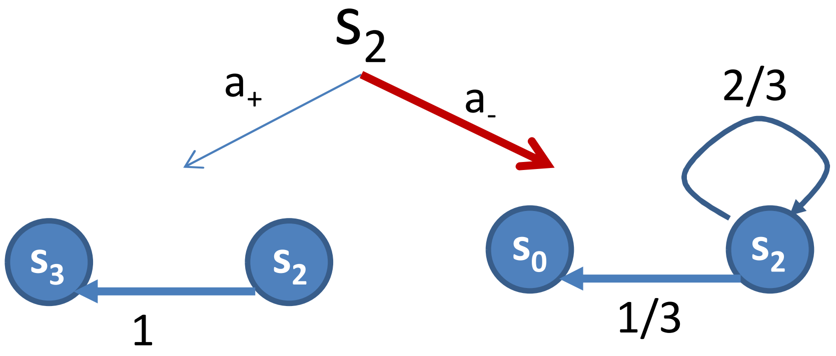
$$V_{s_3} = R_{s_3} + \gamma\left(\frac{2}{3}V_{s_1} + \frac{1}{3}V_{s_3}\right)$$

The Fly MDP: Policy 2 (optimal)

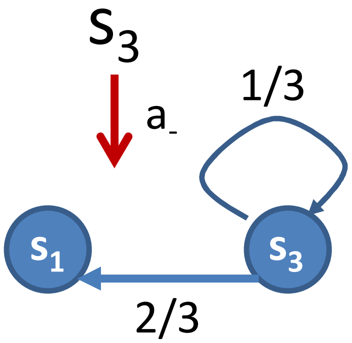




$$V_{s_1} = R_{s_1} + \gamma V_{s_2}$$

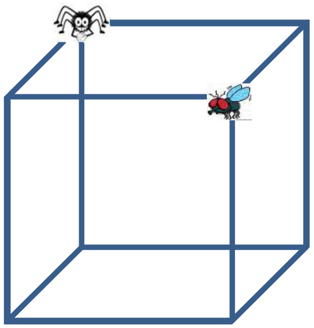


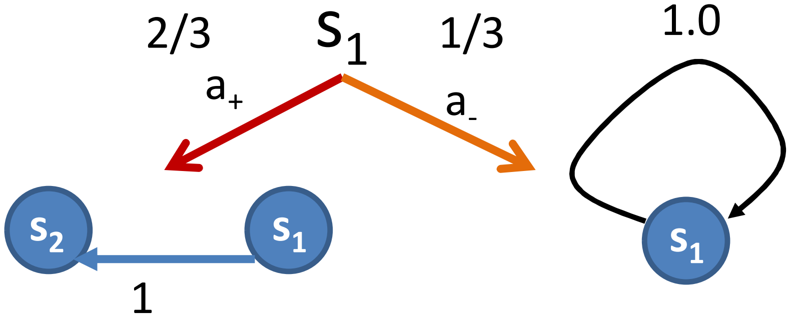
$$V_{s_2} = R_{s_2} + \gamma \left(\frac{1}{3} V_{s_0} + \frac{2}{3} V_{s_2} \right)$$

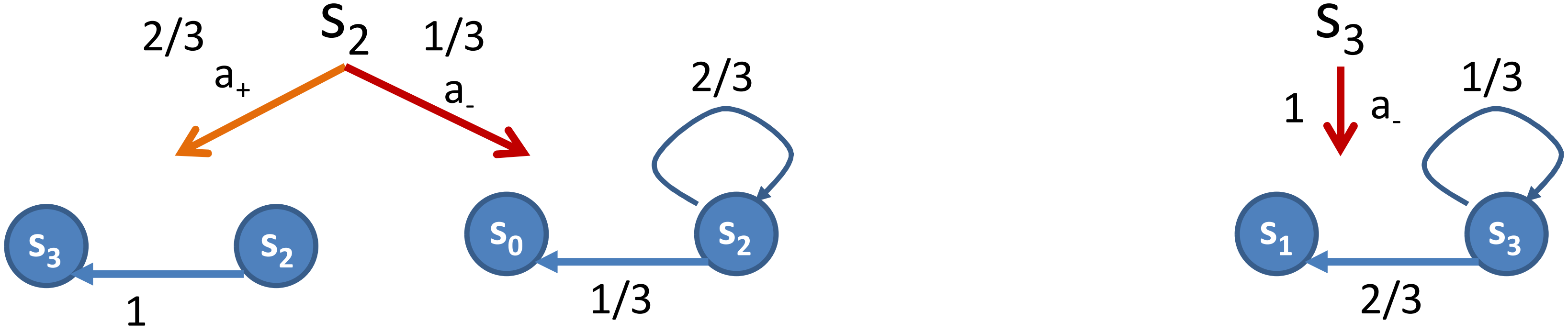


$$V_{s_3} = R_{s_3} + \gamma \left(\frac{2}{3} V_{s_1} + \frac{1}{3} V_{s_3} \right)$$

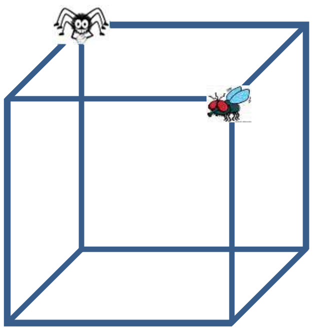
The Fly MDP: Stochastic Policy

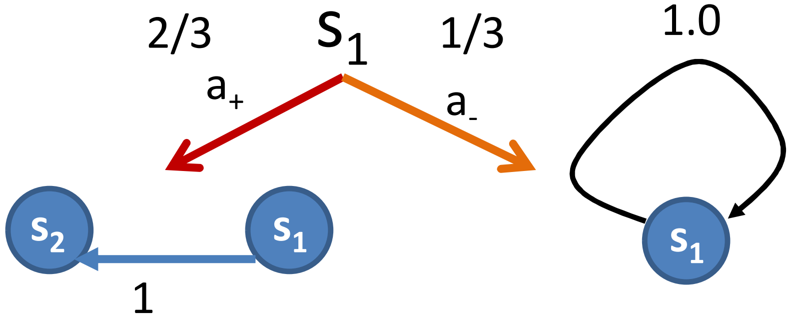




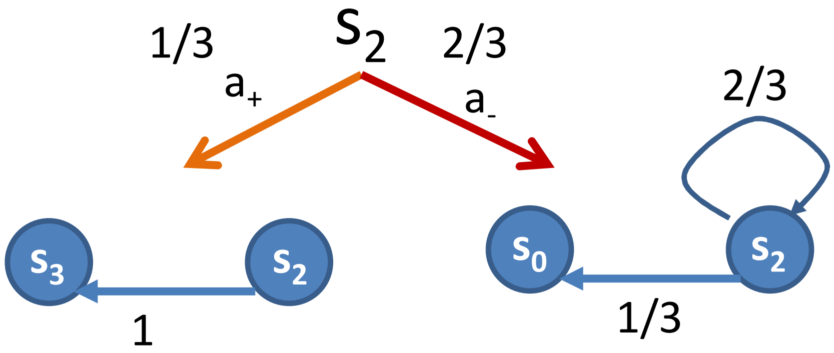


The Fly MDP: Stochastic Policy

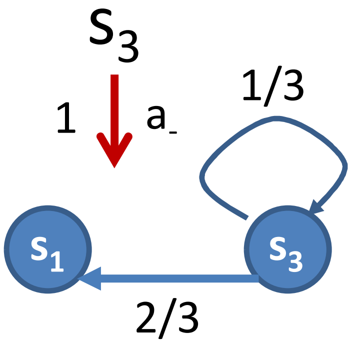




$$V_{s_1} = \frac{2}{3}\left(R_{s_1} + \gamma V_{s_2}\right) + \frac{1}{3}\left(R_{s_1} + \gamma V_{s_1}\right)$$



$$V_{s_2} = \frac{1}{3}\left(R_{s_2} + \gamma V_{s_3}\right) + \frac{2}{3}\left(R_{s_2} + \gamma\left(\frac{1}{3}V_{s_0} + \frac{2}{3}V_{s_2}\right)\right)$$



$$V_{s_3} = R_{s_3} + \gamma\left(\frac{2}{3}V_{s_1} + \frac{1}{3}V_{s_3}\right)$$

The *state value* function of an MDP

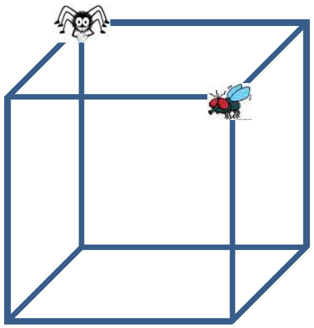
- The *expected return* from any state depends on the policy you follow
- We will index the value of any state by the policy to indicate this

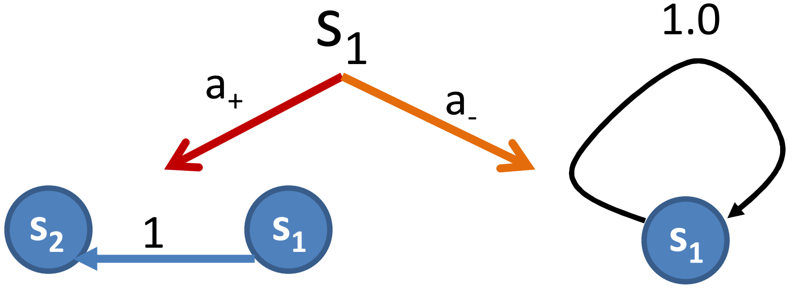
$$v_\pi(s) = \sum_{a \in \mathcal{A}} \pi(a|s) \left(R_s^a + \gamma \sum_{s\prime} P_{s,s\prime}^a v_\pi(s') \right)$$

Bellman Expectation Equation for State Value Functions of an MDP

Note: Although reward was not dependent on action for the fly example, more generally it will be

The action value function of an MDP





$q_{s_1} = R_{s_1} + \gamma v_{s_2}$ $\quad$ $q_{s_1} = R_{s_1} + \gamma v_{s_1}$

- There are different value equations associated with different actions
- So we can actually associate value to ***state action pairs***
- **Note:** The LHS in the equation is the action-specific value at the source state, but the RHS is the overall value of the target states

The *action value* function of an MDP

- The *expected return* from any state under a given policy, when you follow a specific action

$$q_{\pi}(s,a) = R_s^a + \gamma \sum_{s'} P_{s,s'}^a v_{\pi}(s')$$

Bellman Expectation Equation for Action Value Functions of an MDP

All together now

- The Bellman expectation equation for state value function

$$v_\pi(s) = \sum_{a\in\mathcal{A}} \pi(a|s)\left(R_s^a + \gamma \sum_{s\prime} P_{s,s\prime}^a v_\pi(s')\right)$$

- For action value function

$$q_\pi(s,a) = R_s^a + \gamma \sum_{s\prime} P_{s,s\prime}^a v_\pi(s')$$

- Giving you (obviously)

$$v_\pi(s) = \sum_{a\in\mathcal{A}} \pi(a|s) q_\pi(s,a)$$

- And

$$q_\pi(s,a) = R_s^a + \gamma \sum_{s\prime} P_{s,s\prime}^a \sum_{a\prime\in\mathcal{A}} \pi(a|s') q_\pi(s',a')$$

The Bellman Expectation Equations

- The Bellman expectation equation for state value function

$$v_\pi(s) = \sum_{a\in\mathcal{A}} \pi(a|s)\left(R_s^a + \gamma \sum_{s\prime} P_{s,s\prime}^a v_\pi(s')\right)$$

- The Bellman expectation equation for action value function

$$q_\pi(s,a) = R_s^a + \gamma \sum_{s\prime} P_{s,s\prime}^a \sum_{a\prime\in\mathcal{A}} \pi(a|s')q_\pi(s',a')$$

“Computing” the MDP

- Finding the state and/or action value functions for the MDP:
  - Given complete MDP (all transition probabilities $P_{s,s'}^{a}$, expected rewards R_{s}^{a}, and discount γ)
  - and a policy π
  - find all value terms $v_\pi(s)$ and/or $q_\pi(s,a)$

- The Bellman expectation equations are simultaneous equations that can be solved for the value functions
  - Although this will be computationally intractable for very large state spaces

Computing the MDP

$$\mathcal{V}_\pi = \mathcal{R}_\pi + \gamma \mathcal{P}_\pi \mathcal{V}_\pi$$

- Given the expected rewards at every state, the transition probability matrix, the discount factor and the policy:

$$\mathcal{V}_\pi = (\mathrm{I} - \gamma \mathcal{P}_\pi)^{-1} \mathcal{R}_\pi$$

- Matrix inversion $O(N^3)$; intractable for large state spaces

Optimal Policies

- Different policies can result in different value functions
- What is the *optimal* policy?

- The optimal policy is the policy that will maximize the expected total discounted reward at every state:
  $E[G_t|S_t = s]$

$$= E\left[\sum_{k=0}^{\infty} \gamma^k r_{t+k+1} \,|S_t = s\right]$$

Optimal Policies

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$$= E\left[\sum_{k=0}^{\infty} \gamma^k r_{t+k+1} \,|S_t = s\right]$$

  - Recall: why do we consider the *discounted* return, rather than the actual return $\sum_{k=0}^{\infty} r_{t+k+1}$?

Policy Ordering Definition

- A policy π is “better” than a policy π' if the value function under π is greater than or equal to the value function under π' at all states

$$\pi \geq \pi' \ \Rightarrow \ v_{\pi}(s) \geq v_{\pi'}(s) \ \forall s$$

- Under the better policy, you will expect better overall outcome no matter what the current state

The optimal policy theorem

- **Theorem**: For any MDP there exists an optimal policy π_* that is better than or equal to every other policy:

$$\pi_* \geq \pi \quad \forall \pi$$

- **Corollary**: If there are *multiple* optimal policies $\pi_{opt1}, \pi_{opt2}, \ldots$ all of them achieve the same value function

$$v_{\pi_{opti}}(s) = v_*(s) \quad \forall s$$

- All optimal policies achieve the same action value function

$$q_{\pi_{opti}}(s, a) = q_*(s, a) \quad \forall s, a$$

How to find the optimal policy

- For the optimal policy:

$$\pi_*(a|s) = \begin{cases} 1 \ \ for & \underset{a\prime}{\operatorname{argmax}}\, q_*(s, a') \\ 0 & otherwise \end{cases}$$

- Easy to prove
  - For any other policy $\pi,\ \ q_\pi(s, a) \leq q_*(s, a)$

- Knowing the optimal action value function $q_*(s, a)\ \ \forall s, a$ is sufficient to find the optimal policy

The optimal value function

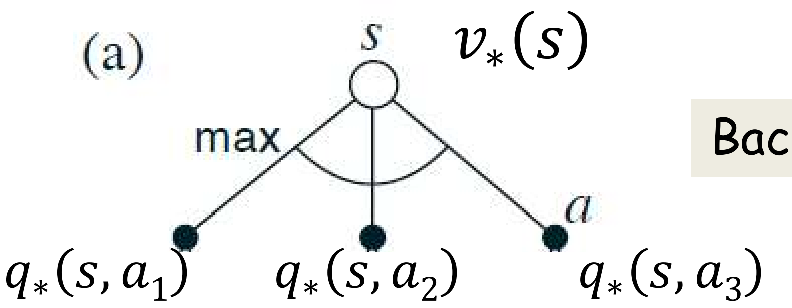
$$\pi_*(a|s) = \begin{cases} 1 \;\; for \quad \underset{a\prime}{\operatorname{argmax}}\, q_*(s, a') \\ \quad\quad 0 \quad\quad otherwise \end{cases}$$

- Which gives us

$$v_*(s) = \max_a q_*(s, a)$$

Pictorially

Figures from Sutton



Backup Diagram

$$v_*(s) = \max_a q_*(s,a)$$

- Blank circles are states, filled dots are state-action pairs

The optimal value function

$$\pi_*(a|s) = \begin{cases} 1 \;\; for & \underset{a\prime}{\operatorname{argmax}}\, q_*(s, a') \\ \quad 0 & otherwise \end{cases}$$

- Which gives us

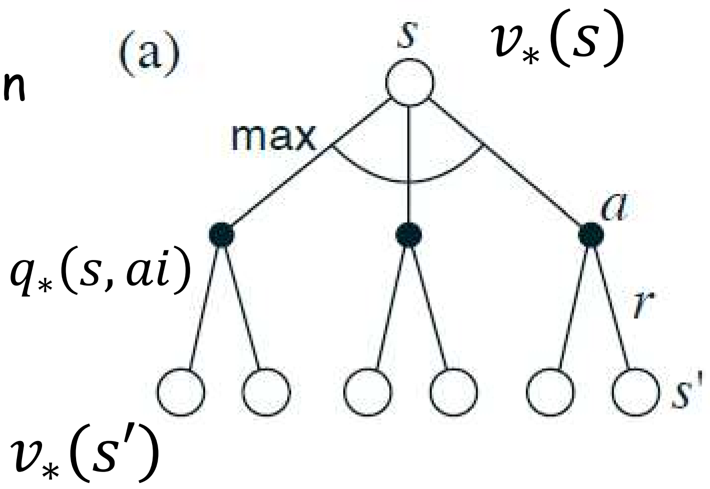
$$v_*(s) = \max_a q_*(s, a)$$

- But, for the optimal policy we also have

$$q_*(s, a) = R_s^a + \gamma \sum_{s\prime} P_{s,s\prime}^a v_*(s')$$

Backup Diagram

Figures from Sutton

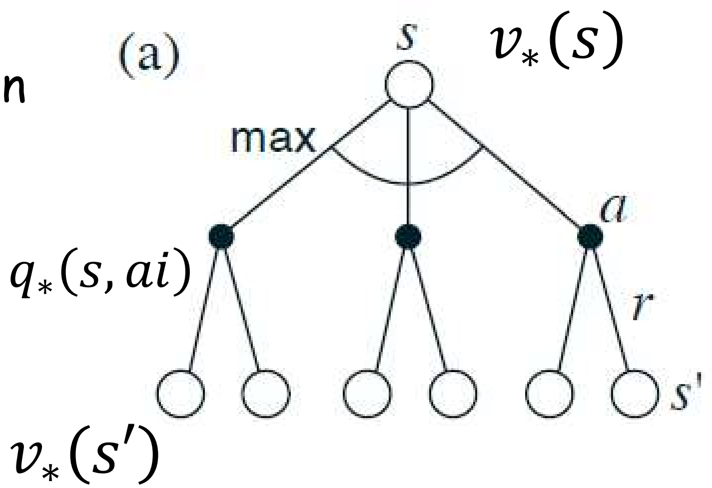


$$v_*(s) = \max_a q_*(s,a)$$

$$q_*(s,a) = R_s^a + \gamma \sum_{s'} P_{s,s'}^a v_*(s')$$

Backup Diagram

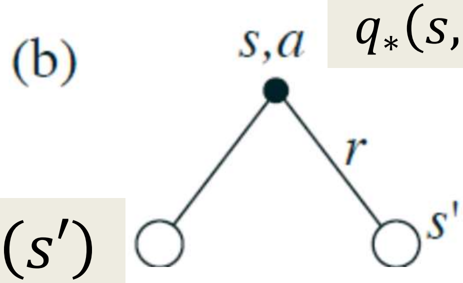
Figures from Sutton



$$v_*(s) = \max_a R_s^a + \gamma \sum_{s'} P_{s,s'}^a v_*(s')$$

Backup Diagram

Figures from Sutton



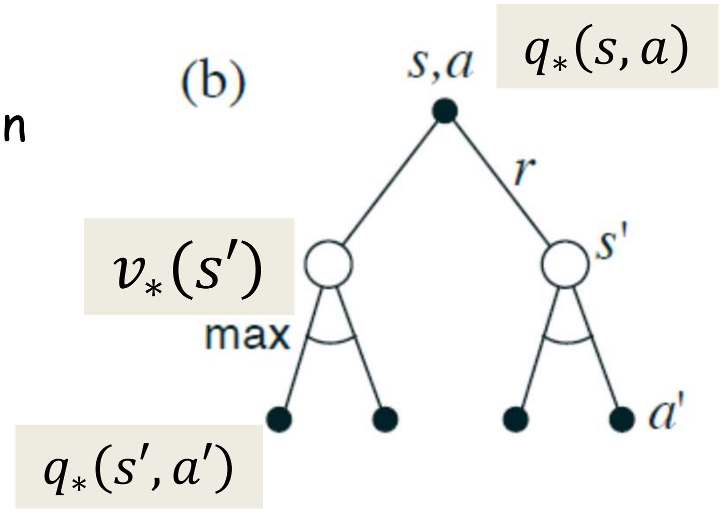
$q_*(s,a)$

$v_*(s')$

$$q_*(s,a) = R_s^a + \gamma \sum_{s'} P_{s,s'}^a v_*(s')$$

Backup Diagram

Figures from Sutton

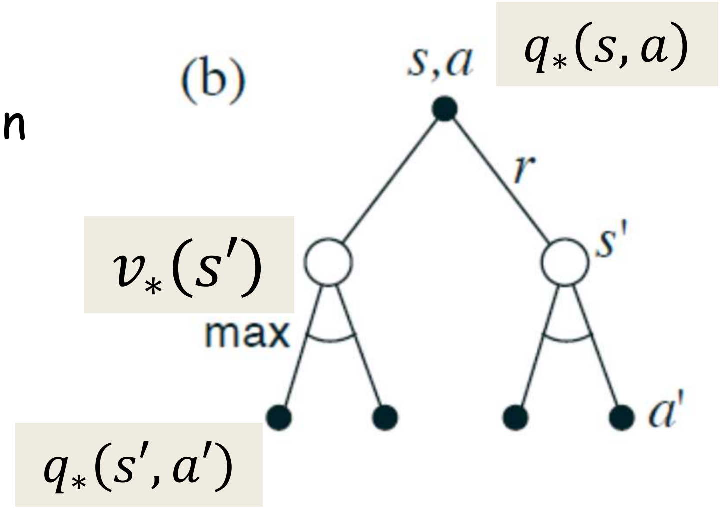


$$q_*(s,a) = R_s^a + \gamma \sum_{s'} P_{s,s'}^a v_*(s')$$

$$v_*(s') = \max_{a'} q_*(s',a')$$

Backup Diagram

Figures from Sutton



$$q_*(s,a) = R_s^a + \gamma \sum_{s'} P_{s,s'}^a \max_{a'} q_*(s',a')$$

Bellman *Optimality* Equations

- Optimal value function equation

$$v_*(s) = \max_a \; R_s^a + \gamma \sum_{s\prime} P_{s,s\prime}^a v_*(s')$$

- Optimal action value equation

$$q_*(s,a) = R_s^a + \gamma \sum_{s\prime} P_{s,s\prime}^a \max_{a\prime} q_*(s',a')$$

Optimality Relationships

- Given the MDP: $\langle \mathcal{S}, \mathcal{P}, \mathcal{A}, \mathcal{R}, \gamma \rangle$
- Given the optimal action value functions, the optimal value function can be found

$$v_*(s) = \max_a q_*(s, a)$$

- Given the optimal value function, the optimal action value function can be found

$$q_*(s, a) = R_s^a + \gamma \sum_{s\prime} P_{s,s\prime}^a v_*(s')$$

- Given the optimal action value function, the optimal policy can be found

$$\pi_*(a|s) = \begin{cases} 1 \;\; for & \operatorname*{argmax}_{a\prime} q_*(s, a') \\ \quad 0 & otherwise \end{cases}$$

“Solving” the MDP

- **Solving the MDP equates to finding the optimal policy $\pi_*(a|s)$**

- Which is equivalent to finding the optimal value function $v_*(s)$
- Or finding the optimal action value function $q_*(s, a)$

- Various solutions will estimate one or the other
  - Value based solutions solve for $v_*(s)$ and $q_*(s, a)$ and derive the optimal policy from them
  - Policy based solutions directly estimate $\pi_*(a|s)$

Solving the Bellman Optimality Equation

- No closed form solutions

- Solutions are iterative
- Given the MDP (Planning):
  - Value iterations
  - Policy iterations

- Not given the MDP (Reinforcement Learning):
  - Q-learning
  - SARSA..

QUESTIONS before we dive?